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Localized modes in type II and heterotic singular Calabi-Yau conformal field theories

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Abstract: We consider type II and heterotic string compactifications on an isolated singularity in the noncompact Gepner model approach. The conifold-type ADE noncompact Calabi-Yau threefolds, as well as the ALE twofolds, are modeled by a tensor product of the $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ Kazama-Suzuki model and an $N=2$ minimal model. Based on the string partition functions on these internal Calabi-Yaus previously obtained by Eguchi and Sugawara, we construct new modular invariant, space-time supersymmetric partition functions for both type II and heterotic string theories, where the GSO projection is performed before the continuous and discrete state contributions are separated. We investigate in detail the massless spectra of the localized modes. In particular, we propose an interesting three generation model, in which each flavor is in the $\mathbf{27} \oplus \mathbf{1}$ representation of E_6 and localized on a four-dimensional space-time residing at the tip of the cigar.

Keywords: Conformal Field Models in String Theory, Superstrings and Heterotic Strings.

Contents

1. Introduction

Building phenomenologically realistic models in string theory is a challenging problem. Among others, one of the most serious obstacles to the construction is the issue of the moduli. Typically, we assume that the background is a product of a four-dimensional Minkowski space and some compact Calabi-Yau manifold. Various parameters characterizing the latter appear as scalar fields in the low-energy effective theory, which are massless until appropriate fluxes and quantum effects are taken into account. The basic question is whether or not, and if so how, the moduli stabilization is realized dynamically. This question is closely linked to the vacuum selection problem. With the recent recognition of the string landscape [1], one might be satisfied if any consistent ultra-violet completion of the Standard Model is obtained, but nothing can guarantee the uniqueness of the solution.

These difficulties stem from the complexity and diversity of compact Calabi-Yau manifolds. Let us suppose that we are given a Calabi-Yau which has only a few, say three, moduli. Then we would not need to worry about the moduli stabilization problem from the beginning. Although there are no such known *compact* Calabi-Yaus, there *are* such *noncompact* ones. A typical example is the ADE series of the ALE manifolds.

After the discovery of D-branes, the use of noncompact local Calabi-Yau manifolds has been common — geometric engineering [2], topological string theory [3] and gauge theory [4] — in all these examples the central focus of the study is the open string. In this paper, in contrast, we use noncompact Calabi-Yaus as the internal sector of conventional closed string compactification in terms of conformal field theory [5], for both type II and heterotic string theories. We consider these superstrings in a four- (and also six-) dimensional Minkowski space with some internal noncompact conifold-like threefold (ALE twofold) of the ADE type, where the internal part is described [6, 7] by a tensor product of an $N = 2$ minimal model with level $k_{\rm min} = 0, 1, 2\ldots$ and the noncompact coset $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ Kazama-Suzuki model with correlated level κ.

We present a compact expression for space-time supersymmetric, modular invariant partition functions consisting not only of contributions from the continuous (principal unitary) series representations of the mother $\mathrm{SL}(2,\mathbf{R})$ Lie algebra, but also of those from the discrete series representations. This new space-time supersymmetric partition function is an improvement of the earlier results in noncritical super strings or "noncompact" Gepner models [8 − 12], and owes much to the recent construction of modular invariants for the internal noncompact Calabi-Yau CFTs by Eguchi and Sugawara [13]. We will show, by using the character decomposition technique [13 − 15], there are massless matter supermultiplets coming from the discrete series, the number of which can be small depending on the value of the level $k_{\rm min}$. In particular, if we consider the $E_8 \times E_8$ heterotic string compactification[1] for $k_{\rm min} = 3$, we will find precisely three generations of $N = 1$ chiral multiplets

[1] We should note that in the heterotic case our construction is closely related to the "heterotic coset models" [16] studied earlier because an $N = 2$ minimal model is realized [17] as an $\mathrm{SU}(2)/\mathrm{U}(1)$ coset theory.

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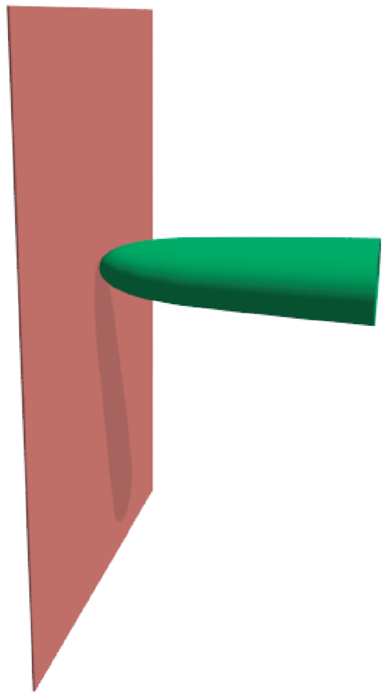

Figure 1: The schematic picture.

in the $\mathbf{10} \oplus \mathbf{16} \oplus \mathbf{1} \oplus \mathbf{1}$ of SO(10) or $\mathbf{27} \oplus \mathbf{1}$ of E_6. Since the discrete series representations in the $\mathrm{SL}(2, \mathbf{R})/\mathrm{U}(1)$ gauged WZW model are known to be the modes localized [18] near the tip of the "cigar" [19], these three flavors can move only in the four-dimensional Minkowski directions, and hence are trapped on some four-manifold at the tip of the cigar. The schematic picture is shown in figure 1.

This "brane" is not a D-brane; the localized modes are those of closed strings, which exist even in heterotic string theories. In fact, these modes can be regarded as the position moduli of NS5-branes. Indeed, in the six-dimensional analysis with an ALE manifold, we will find [13] precisely as many massless supermultiplets in the discrete spectrum as the number of two-cycles, which are $D = 6$ *nonchiral* $N = 2$ multiplets (including vectors) in the type IIA case and *chiral* ones (including anti-selfdual tensors) in the type IIB case. They are opposite to the zero modes appearing on NS5-branes [20] in agreement with the T-duality [7, 21] between the NS5-brane and the ADE singularity. In the IIB case, the S-dual version was used in the past to explain [22] the nonperturbative gauge symmetry enhancement near the singularity [23] in terms of D-branes.[2] In the four-dimensional case, relations between a deformed conifold and a system of intersecting NS5-branes are also known [22, 28]. We would like to emphasize that to even see geometric moduli of a noncompact Calabi-Yau as massless modes in a modular invariant CFT partition function has been a nontrivial problem.

The dynamics on NS5-branes in the framework of the CHS model [30, 20] was much

[2]The nonperturbative "W-bosons" cannot be seen in our closed string CFT partition functions. They can, however, be analyzed [24 – 27] in the boundary Liouville CFT [29], which we do not consider in this paper. We would like to thank Y. Sugawara for discussion on this point.

studied as “Little String Theories” (LSTs) [31]. They are basically non-critical superstring theories [32] coupled to some compact CFT, which is a supersymmetric SU(2) WZW model for NS5-branes. In analogy to the AdS/CFT correspondence [33], it has been proposed that their vanishing string coupling ($g_s \to 0$) limit (and hence the decoupling gravity limit) has some holographic dual theory on the boundary at the weakly coupled linear dilaton region (the“mouth” of the throat). To avoid the strong-coupling singularity far down the throat, we need a regularization in the bulk theory. There are two known ways: The first is the so-called double-scaled Little String Theory [34, 35], that is, a particular limit of LST where the weak string coupling limit and the limit of collapsing areas of the homology cycles are taken in a correlated manner. In this limit, the physics depends only on a particular combination of the coupling constant and a deformation parameter, and the scaled theory can be weakly coupled. The second is to replace the linear-dilaton cylinder geometry with the cigar geometry [6, 7]. Later it was shown that these two are dual to each other [34, 36].

In fact, the link between the NS5-brane and the two-dimensional black hole goes back to the work by Gidding and Strominger (GM) [37] in 1991, where a similar double-scaling (and, at the same time, extremal) limit of a family of type II and heterotic *non-extremal* black five-brane solutions was considered to observe that the resulting geometry was a product of a $(1 + 1)$-dimensional black hole, an S^3 and a five-dimensional Euclidean space. The CFT description of this geometry is very close to ours; although it is not exactly the same (because, for instance, we consider a Euclidean black hole), it is at least suggestive. In this GM’s double-scaling limit, despite the $g_s \to 0$ limit taken there, the graviton, dilaton and other backgrounds do not disappear but are still present in the final geometry with nontrivial, though finite, profiles in the whole space-time.[3]

As we mentioned above, our new partition functions are constructed based on the ones for internal noncompact Calabi-Yaus obtained by [13]. Roughly speaking, what we do is to couple the noncompact Calabi-Yau CFT to that for the flat Minkowski space and perform a suitable GSO projection *before* the contributions from the continuous and discrete series representations are separated. The states in the latter class of $\mathrm{SL}(2, \mathbf{R})$ representations will be called the “discrete states” in short. It turns out that the formulas are simple and similar in their form to those for the partition functions containing only the continuous representations obtained previously. One of the virtues of our formulas is that the couplings of the discrete states for the Calabi-Yau to states for the flat Minkowski space are automatically consistent with the modular invariance of the continuous sector. Another advantage is that we can straightforwardly extend the type II analysis to heterotic strings by using the heterotic conversion procedure [5] of modular invariant partition functions. As we noted above, we can construct an interesting three generation model, in which each flavor consists of a **27** and a singlet of E_6 and is localized on a four-dimensional space-time. Thus this ($k_{\min} = 3$) model may offer a viable alternative string model for the E_6 unification [40].

Gravity and gauge fields are not localized; they are (apparently) massive due to the Liouville energy and propagate into the bulk. They correspond to the continuous series

[3]With the standard embedding, this GM’s double scaling also regularizes the small instanton singularity [38, 39] for heterotic five-branes.

representations. But still, we expect that the three generation model above will be useful for studying issues of flavors. While any particular phenomenological realization on a compact Calabi-Yau cannot be unique, singularities occur *universally* in the moduli space of any compact Calabi-Yau manifold. We hope we can capture some universal physics near the singularity by studying the localized modes in the conformal field theory.

This is a more detailed version of [41], in which the summary of results presented here was already announced. The plan of this paper is as follows. In section 2, we review the basics of representation theory of the affine $\mathrm{SL}(2,\mathbf{R})$ and the associated $N=2$ superconformal algebras. In section 3, we also review the previous constructions of modular invariant partition functions in the noncompact Gepner model approach, which consists of contributions from only the continuous (principal unitary) series representations. In section 4, we construct new space-time supersymmetric, modular invariant partition functions on the ADE generalization of conifolds, for both type II and heterotic string theories. In section 5, we describe the detail of how to separate the discrete series contributions from the new partition functions, and examine the spectrum. In particular, we propose the $k_{\min}=3$ three generation model mentioned above. Section 6 is devoted to examples. In section 7, we briefly discuss the generalization to the six-dimensional space-time with the ordinary ALE manifolds. Finally, we conclude this paper with a summary and discussion, which are given in section 8. Appendix A contains basic definitions of theta functions and characters, and their identities. In appendix B we collect useful formulas related the functions $F_{l,2r}(\tau,z)$ and $\hat{F}_{l,2r}(\tau,z)$ we use in the text, which are important building blocks in the construction of the partition functions. Appendix C is a review of the heterotic conversion procedure of Gepner. Finally, in appendix D we give a proof of the regularization formula of [14].

2. SL(2, R) paraferemions and $N = 2$ superconformal algebra

In this section we review the relation between the affine $\mathrm{SL}(2,\mathbf{R})$ Kac-Moody and $N=2$ superconformal algebras based on the $\mathrm{SL}(2,\mathbf{R})$ parafermion construction of [42]. This is relevant for our discussion because we construct a model by using the $N=2$ representations while the "localization of modes" is a concept that has emerged in the $\mathrm{SL}(2,\mathbf{R})$ ones.

2.1 Free field realizations

The $\mathrm{SL}(2,\mathbf{R})$ Kac-Moody currents of level κ are realized as follows:

$$J^3(z) = i\sqrt{\frac{\kappa}{2}}\partial\phi, \tag{2.1}$$

$$J^{\pm}(z) = i\left(\sqrt{\frac{\kappa}{2}}\partial\theta \pm i\sqrt{\frac{\kappa-2}{2}}\partial\rho\right)\exp\left(\pm i\sqrt{\frac{2}{\kappa}}(\theta-\phi)\right), \tag{2.2}$$

where $\rho(z)$, $\theta(z)$ and $\phi(z)$ are free fields satisfying the following OPEs:

$$\rho(z)\rho(w) \sim -\log(z-w), \tag{2.3}$$

$$\theta(z)\theta(w) \sim -\log(z-w), \tag{2.4}$$

$$\phi(z)\phi(w) \sim +\log(z-w). \tag{2.5}$$

The energy-momentum tensor $T^{\mathrm{SL}(2,\mathbf{R})}(z)$ is given by

$$T^{\mathrm{SL}(2,\mathbf{R})}(z) = -\frac{1}{2}(\partial\rho)^2 + \frac{1}{\sqrt{2(\kappa-2)}}\partial^2\rho - \frac{1}{2}(\partial\theta)^2 + \frac{1}{2}(\partial\phi)^2, \tag{2.6}$$

which has a central charge

$$c_{\mathrm{SL}(2,\mathbf{R})} = \frac{3\kappa}{\kappa-2}. \tag{2.7}$$

The $\mathrm{SL}(2,\mathbf{R})$ parafermions $\psi^{\pm}(z)$ [42] are fundamental fields in the $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ coset conformal field theory. They are written in terms of the free fields as

$$\psi^{\pm}(z) = i\left(\sqrt{\frac{1}{2}}\partial\theta \pm i\sqrt{\frac{\kappa-2}{2\kappa}}\partial\rho\right)\exp\left(\pm i\sqrt{\frac{2}{\kappa}}\theta\right). \tag{2.8}$$

Using these fields with another free boson φ [43] satisfying

$$\varphi(z)\varphi(w) \sim -\log(z-w), \tag{2.9}$$

a set of $N=2$ superconformal currents are realized as follows:

$$T^{N=2}(z) = -\frac{1}{2}(\partial\rho)^2 + \frac{1}{\sqrt{2(\kappa-2)}}\partial^2\rho - \frac{1}{2}(\partial\theta)^2 - \frac{1}{2}(\partial\varphi)^2, \tag{2.10}$$

$$G^{\pm}(z) = \sqrt{\frac{2\kappa}{\kappa-2}}\psi^{\pm}(z)\exp\left(\pm i\sqrt{\frac{\kappa-2}{\kappa}}\varphi\right), \tag{2.11}$$

$$J^{N=2}(z) = i\sqrt{\frac{\kappa}{\kappa-2}}\partial\varphi. \tag{2.12}$$

The central charge $c_{N=2}$ is the same as $c_{\mathrm{SL}(2,\mathbf{R})}$:

$$c_{N=2} = \frac{3\kappa}{\kappa-2}. \tag{2.13}$$

2.2 Unitary representations of the $\mathrm{SL}(2,\mathbf{R})$ and $N=2$ superconformal algebras

A unitary module ("Fock space") of the affine $\mathrm{SL}(2,\mathbf{R})$ Kac-Moody algebra necessarily contains a unitary (non-affine) $\mathrm{SL}(2,\mathbf{R})$ algebra module at the lowest $L_0^{\mathrm{SL}(2,\mathbf{R})}$ level. The states in the module are labeled by the eigenvalues of J_0^3 and

$$\boldsymbol{J} = \frac{1}{2}(J_0^+J_0^- + J_0^-J_0^+) - (J_0^3)^2. \tag{2.14}$$

Let us denote[4] such an eigenstate by

$$|l, m+\epsilon> \quad (m \in \mathbf{Z}, \quad 0 \le \epsilon < 1), \tag{2.15}$$

[4] We consider the universal cover of $\mathrm{SL}(2,\mathbf{R})$.

where

$$J_0^3|l,m+\epsilon> = (m+\epsilon)|l,m+\epsilon>, \tag{2.16}$$

$$J_0^{\pm}|l,m+\epsilon> = (m+\epsilon\pm l)|l,m+\epsilon\pm 1>, \tag{2.17}$$

$$\boldsymbol{J}|l,m+\epsilon> = -l(l-1)|l,m+\epsilon>. \tag{2.18}$$

A state $|l,m+\epsilon>$ corresponds to a vertex operator

$$|l,m+\epsilon> \rightarrow e^{\sqrt{\frac{2}{\kappa-2}}l\rho+i\sqrt{\frac{2}{\kappa}}(m+\epsilon)(\theta-\phi)} \tag{2.19}$$

in the free field realization of the affine SL$(2,\mathbf{R})$ Kac-Moody algebra. It has a conformal weight

$$L_0^{\mathrm{SL}(2,\mathbf{R})} = -\frac{l^2-l}{\kappa-2}, \tag{2.20}$$

$$J_0^3 = m+\epsilon. \tag{2.21}$$

The corresponding $N=2$ vertex operator is then given by

$$\rightarrow e^{\sqrt{\frac{2}{\kappa-2}}l\rho+i\sqrt{\frac{2}{\kappa}}(m+\epsilon)\theta+i\sqrt{\frac{2}{\kappa}}(m+\epsilon)\varphi}, \tag{2.22}$$

which is a primary field with eigenvalues

$$L_0^{N=2} = \frac{-(l^2-l)+(m+\epsilon)^2}{\kappa-2} \quad (\equiv h), \tag{2.23}$$

$$J_0^{N=2} = \frac{-2(m+\epsilon)}{\kappa-2} \quad (\equiv Q). \tag{2.24}$$

The point is that [42] each individual state $|l,m+\epsilon>$ in a unitary representation of the non-affine SL$(2,\mathbf{R})$ algebra corresponds to a unitary representation of the $N=2$ superconformal algebra. We will consider each class of representations separately.

(i) The principal unitary series (The "continuous series"). The representation space in this class consists of a set of states

$$\{|l,m+\epsilon> \ | \ m\in\mathbf{Z}\} \tag{2.25}$$

for some $l=\frac{1}{2}+ip$, $p\in\mathbf{R}$ and $0\leq\epsilon<1$. There is neither upper nor lower J_0^3 bound in the states. The corresponding $N=2$ representation has

$$h = \frac{1}{\kappa-2}\left(p^2+\frac{1}{4}+(m+\epsilon)^2\right), \tag{2.26}$$

$$Q = -\frac{2(m+\epsilon)}{\kappa-2}. \tag{2.27}$$

Eliminating $m+\epsilon$, we obtain a family of parabola

$$h = \frac{\kappa-2}{4}Q^2+\frac{1}{\kappa-2}\left(p^2+\frac{1}{4}\right) \tag{2.28}$$

labeled by $p\in\mathbf{R}$ on the (h,Q)-plane. They are shown in blue in figure 2. Throughout this paper, the term "continuous series" will refer to this class of representations.

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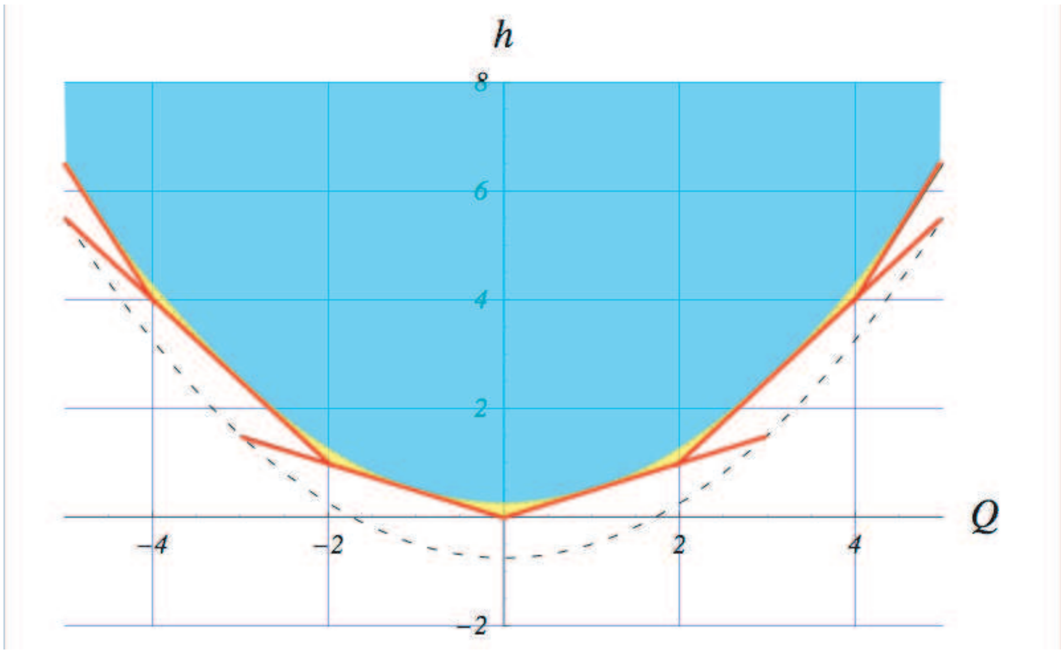

Figure 2: The unitary region of the $N = 2$ superconformal algebra [44] ($c = 9$, NS sector).

(ii) The discrete series $\mathcal{D}_n^+$ ($n = 0, 1, \ldots$). The representation space consists of states $|l, m + \epsilon >$ such that, for a given n,

$$l = n + \epsilon, \tag{2.29}$$

$$m + \epsilon = n + \epsilon + r \quad (r = 0, 1, 2, \ldots). \tag{2.30}$$

The representation $\mathcal{D}_n^+$ has a lowest-J_0^3 state

$$|l, m + \epsilon > = |n + \epsilon, n + \epsilon > . \tag{2.31}$$

The values of h and Q of the corresponding $N = 2$ representations are

$$h = \frac{1}{\kappa - 2}\left((2r + 1)(n + \epsilon) + r^2\right), \tag{2.32}$$

$$Q = -\frac{2}{\kappa - 2}(n + \epsilon + r). \tag{2.33}$$

Eliminating $n + \epsilon$ from above, we obtain

$$h = -\left(r + \frac{1}{2}\right) Q - \frac{1}{\kappa - 2}\left(\left(r + \frac{1}{2}\right)^2 - \frac{1}{4}\right) \quad (r = 0, 1, 2, \ldots). \tag{2.34}$$

They are precisely the left half of the family of segments which bound the unitary region on the (h, Q)-plane. They are shown in red lines in figure 2.

(iii) The discrete series $\mathcal{D}_n^-$ ($n = 1, 2, \ldots$). The representation space of $\mathcal{D}_n^-$ consists of $|l, m + \epsilon >$ such that

$$l = n - \epsilon, \tag{2.35}$$

$$m + \epsilon = -n + \epsilon - r \quad (r = 0, 1, 2, \ldots), \tag{2.36}$$

among which

$$|l, m + \epsilon > = |n - \epsilon, -n + \epsilon > \tag{2.37}$$

is the highest-J_0^3 state. The corresponding h and Q are similarly

$$h = \frac{1}{\kappa - 2}\left((2r + 1)(n - \epsilon) + r^2\right), \tag{2.38}$$

$$Q = +\frac{2}{\kappa - 2}(n - \epsilon + r), \tag{2.39}$$

and therefore

$$h = +\left(r + \frac{1}{2}\right) Q - \frac{1}{\kappa - 2}\left(\left(r + \frac{1}{2}\right)^2 - \frac{1}{4}\right) \quad (r = 0, 1, 2, \ldots). \tag{2.40}$$

They are the right half of the family of the segments. They are also shown in red lines.

(iv) The complimentary series. The complimentary series is similar to the principal unitary series, but in this case l and ϵ satisfy $(0 \leq \epsilon < 1)$

$$-\epsilon(\epsilon - 1) \leq -l(l - 1) < \frac{1}{4}. \tag{2.41}$$

The upper bound of $\boldsymbol{J}$

$$-l(l - 1) = \frac{1}{4} \tag{2.42}$$

coincides with the principal unitary series with $p = 0$, while the lower bound

$$-l(l - 1) = -\epsilon(\epsilon - 1) \tag{2.43}$$

the $N = 2$ representations which arise from the states in $\mathcal{D}^+_{n=0} \cup \mathcal{D}^-_{n=1}$. Therefore, the complimentary series fill the gap between the paraboloid of the $p = 0$ principal unitary series and the polygonal boundary of the discrete series representations. They are shown in figure 2 as the narrow yellow region between the blue area and the red segments.

Although this class of representations is much like the principal unitary series with continuous spectra, they do not arise in our model.

(v) The trivial (identity) representation. It gives rise to the identity representation

$$h = 0, \quad Q = 0 \tag{2.44}$$

of the $N = 2$ superconformal algebra.

3. Noncompact Gepner models

In this section we review the old noncompact Gepner model constructions and address their issues.

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3.1 Modular invariant partition functions

In usual Gepner models [5], one uses a tensor product of $N = 2$ minimal superconformal field theories so that their central charge add up to nine for compactification to four dimensions. They are subject to an orbifold projection, which is implemented by taking an alternating sum over shifted indices of theta functions in the minimal characters. This is called the "β-method" [45], with which one can achieve both an integral total U(1) charge and modular invariance. In modern terminology, it is equivalent to consider spectral flow orbits with respect to the $N = 2$ U(1) charge.

The $N = 2$ minimal models are labeled by a nonnegative integer level $k_{\rm min}$ and have central charges $c_{\rm min} = \frac{3k_{\rm min}}{k_{\rm min}+2}$, which do not exceed three. Therefore, we need at least four minimal models to have nine. In [8], an attempt was made to construct a supersymmetric modular invariant partition function by using $c = 9$ characters directly, with no minimal models. Such representations are necessarily nonminimal ones. A generic (and hence nonminimal) $N = 2$ character of a representation with a highest weight $L_0^{N=2} = h$, $J_0^{N=2} = Q$ is given by [46]

$$\mathrm{Tr}_{\rm NS}\, q^{L_0^{N=2}} y^{J_0^{N=2}} = q^{h+\frac{1}{8}} y^Q \frac{\vartheta_3(\tau, z)}{\eta^3(\tau)} \tag{3.1}$$

for the NS sector, and

$$\mathrm{Tr}_{\rm R}\, q^{L_0^{N=2}} y^{J_0^{N=2}} = q^{h+\frac{1}{8}} y^Q \frac{\vartheta_2(\tau, z)}{\eta^3(\tau)} \tag{3.2}$$

for the R sector. The definitions of theta functions, as well as other functions used below, are summarized in appendix A. To improve the modular property of these nonminimal characters and construct a modular invariant, the following two ideas were considered [8]: The first is to gather infinitely many generic representations with different U(1) charges aligned on a lattice, so that the infinite sum produces another theta function. The second idea is to integrate over the continuous spectrum of generic characters with a Gaussian weight with respect to the Liouville momentum p (2.28). For reasons that will be explained below, we use level-1 theta functions for the first idea. Then, taking into account the Gaussian integration, we have roughly

$$\frac{1}{\sqrt{\tau_2}} \left| \frac{\Theta_{*,1}\Theta_{*,2}}{\eta^3} \right|^2 , \tag{3.3}$$

which has modular weight $(0, 0)$.

Next, to achieve spacetime supersymmetry, we need a GSO projection. We have two level-2 theta functions, one from the complex fermion for the flat two-dimensional transverse space and the other from the $N = 2$ character above. The β-method tells us how to construct good combinations of theta functions. That is, we consider an alternating summation of the form

$$\sum_\nu (-1)^\nu \Theta_{m+\beta_0\nu,k} \Theta_{s_1+\beta_1\nu,2} \Theta_{s_2+\beta_2\nu,2} \tag{3.4}$$

with the "β-conditions" [5]:

$$\frac{\beta_0^2}{2k} + \frac{\beta_1^2}{4} + \frac{\beta_2^2}{4} = 1, \tag{3.5}$$

$$\frac{\beta_0 m}{2k} + \frac{\beta_1 s_1}{4} + \frac{\beta_2 s_2}{4} = \frac{1}{2}. \tag{3.6}$$

The solution is $k = 1$, $(\beta_0, \beta_1, \beta_2) = (1, 1, 1)$ and $(m, s_1, s_2) = (1, 0, 0)$ or $(0, 0, 2)$. Indeed, if we define

$$\begin{aligned}
\Lambda_1(\tau, z) &\equiv 2 \sum_{\nu \in \mathbf{Z}_4} (-1)^\nu \Theta_{1+\nu,1}(\tau, 2z)\Theta_{\nu,2}(\tau, z)\Theta_{\nu,2}(\tau, z) \\
&= \Theta_{1,1}(\tau, 2z)\Big(\vartheta_3^2(\tau, z) + \vartheta_4^2(\tau, z)\Big) - \Theta_{0,1}(\tau, 2z)\Big(\vartheta_2^2(\tau, z) + \tilde{\vartheta}_1^2(\tau, z)\Big),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\Lambda_2(\tau, z) &\equiv 2 \sum_{\nu \in \mathbf{Z}_4} (-1)^\nu \Theta_{\nu,1}(\tau, 2z)\Theta_{\nu,2}(\tau, z)\Theta_{2+\nu,2}(\tau, z) \\
&= \Theta_{0,1}(\tau, 2z)\Big(\vartheta_3^2(\tau, z) - \vartheta_4^2(\tau, z)\Big) - \Theta_{1,1}(\tau, 2z)\Big(\vartheta_2^2(\tau, z) - \tilde{\vartheta}_1^2(\tau, z)\Big),
\end{aligned} \tag{3.8}$$

then their modular transformations are [8]

$$\begin{aligned}
\Lambda_1(\tau + 1, 0) &= i\ \Lambda_1(\tau, 0), \\
\Lambda_2(\tau + 1, 0) &= -\Lambda_2(\tau, 0),
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\Lambda_1\left(-\frac{1}{\tau}, 0\right) &= \frac{\tau^{3/2} e^{-\frac{3\pi i}{4}}}{\sqrt{2}} \left(-\Lambda_1(\tau, 0) + \Lambda_2(\tau, 0)\right), \\
\Lambda_2\left(-\frac{1}{\tau}, 0\right) &= \frac{\tau^{3/2} e^{-\frac{3\pi i}{4}}}{\sqrt{2}} \left(+\Lambda_1(\tau, 0) + \Lambda_2(\tau, 0)\right).
\end{aligned} \tag{3.10}$$

Therefore

$$\frac{|\Lambda_1(\tau, 0)|^2 + |\Lambda_2(\tau, 0)|^2}{|\eta^3(\tau)|^2} \tag{3.11}$$

is modular invariant. In fact, the functions $\Lambda_1(\tau, z)$ and $\Lambda_2(\tau, z)$ vanishes identically for whatever value of z, and hence play the role of Jacobi's quartic identity in the ordinary ten-dimensional critical superstring theories. Λ_1 was used long time ago [47], and Λ_2 was derived in [8] by a modular transformation. It was clarified [13] that these (identically zero) functions, as well as more general combinations of theta functions and the $N = 2$ minimal characters, were derived from Jacobi's identity through compositions of theta functions. Using (3.11), we can write a modular invariant

$$Z = \int \frac{d\tau d\overline{\tau}}{\mathrm{Im}\tau} (\mathrm{Im}\tau)^{-2} |\eta(\tau)|^{-4} (\mathrm{Im}\tau)^{-\frac{1}{2}} |\eta(\tau)|^{-2} \frac{|\Lambda_1(\tau, 0)|^2 + |\Lambda_2(\tau, 0)|^2}{|\eta^3(\tau)|^2}, \tag{3.12}$$

where the factor $(\mathrm{Im}\tau)^{-\frac{1}{2}}$ comes from the Liouville momentum integration, and $|\eta(\tau)|^{-2}$ from the transverse fermions. The transverse fermion thetas are contained in Λ's.

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3.2 The link to singular Calabi-Yaus

The modular invariant (3.12) is regarded as a partition function of type II strings "compactified" on the conifold [6, 7]. The defining equation of the (deformed) conifold is [48]

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu \tag{3.13}$$

in $\mathbf{C}^4$ with a deformation constant μ. If we view (3.13) as an equation in inhomogeneous coordinates of some weighted projective space, we may recover the homogeneous expression

$$-\mu z_0^{-1} + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0, \tag{3.14}$$

where the negative power of z_0 is determined by the Calabi-Yau condition. According to the well-known relation between the Landau-Ginzburg potential and the level of the minimal model [49, 50], the first term suggests that the (deformed) conifold is described by the level-$(-1-2=-3)$ minimal model, which was interpreted [51, 6, 7][5],[6] as the level-$(+3)$ $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ Kazama-Suzuki model [53] which has $c=9$. Prior to this, the circle of connections between the the topological string near the conifold limit, twisted $\mathrm{SU}(2)/\mathrm{U}(1)$ coset at level-(-3), the $c=1$ string at the self-dual radius and matrix models had been noted [51, 54 – 58].

The ADE singularity of a Calabi-Yau twofold [59] can be considered similarly [7]. The X_n singularity ($X = A, D$ or E) is defined by an algebraic equation

$$W_{X_n}(z_1, z_2, z_3) = 0 \tag{3.15}$$

in $\mathbf{C}^3$, where

$$W_{A_n}(z_1, z_2, z_3) \equiv z_1^{n+1} + z_2^2 + z_3^2, \tag{3.16}$$
$$W_{D_n}(z_1, z_2, z_3) \equiv z_1^{n-1} + z_1 z_2^2 + z_3^2, \tag{3.17}$$
$$W_{E_6}(z_1, z_2, z_3) \equiv z_1^4 + z_2^3 + z_3^2, \tag{3.18}$$
$$W_{E_7}(z_1, z_2, z_3) \equiv z_1^3 z_2 + z_2^3 + z_3^2, \tag{3.19}$$
$$W_{E_6}(z_1, z_2, z_3) \equiv z_1^5 + z_2^3 + z_3^2. \tag{3.20}$$

The singularity equation (3.15) is similarly deformed to

$$W_{X_n}(z_1, z_2, z_3) = \mu z_0^{-h^\vee(X_n)}, \tag{3.21}$$

where $h^\vee(X_n)$ is the (dual) Coxeter number of the Lie algebras $h^\vee = n+1, 2(n-1), 12, 18$ and 30 for $X_n = A_n$, D_n, E_6, E_7 and E_8, respectively. Again, (3.21) is understood as an equation in some weighted projective space specified by the weight of z_0, which is determined by the Calabi-Yau condition. (3.21) indicates that the deformed ADE singularities are described by the $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ Kazama-Suzuki model of level $(h^\vee + 2)$ coupled to the

[5]In these references the Kac-Moody level was denoted by k. Note that we denote this by κ while we differently use k meaning $k = \kappa - 2$ in this paper, following the notation of [13].

[6]See however [52] for possible subtleties in the connection between noncompact CFTs and Calabi-Yau geometries.

level $(h^\vee - 2)$ $N = 2$ minimal model of the corresponding modular invariant type. It was also argued by using the character identity (A.19) [5] that the ADE singularity was T-dual to the NS5-brane.

The corresponding modular invariant partition functions for type II strings "compactified" on these noncompact manifolds, as well as on similar ADE generalizations of the conifold, were constructed in [13] by using the generic noncompact $N = 2$ characters, where it was revealed that the relevant spectral flow orbits which constituted the partition function were actually obtained in a unified way by composing the theta functions in Jacobi's identity. Namely, in the twofold case, the authors of [13] defined the functions

$$\begin{aligned} F_l(\tau,z) &\equiv \frac{1}{2}\chi_l^{(k_{\min})}(\tau,0)\left(\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 + \tilde{\vartheta}_1^4\right)(\tau,z) \\ &= \sum_{\nu\in\mathbf{Z}_4}(-1)^\nu \sum_{m\in\mathbf{Z}_{2(k_{\min}+2)}} \chi_m^{l,\nu}(\tau,-z) \sum_{\substack{\nu_0,\,\nu_1,\,\nu_2\,\in\,\mathbf{Z}_2 \\ \nu_0+\nu_1+\nu_2 \\ \equiv 1(\mathrm{mod}2)}} \Theta_{2\nu_0+\nu,2}(\tau,z)\Theta_{2\nu_1+\nu,2}(\tau,z) \\ &\qquad \cdot\Theta_{2\nu_2+\nu,2}(\tau,z)\Theta_{m,k_{\min}+2}\left(\tau,\frac{2z}{k_{\min}+2}\right) \end{aligned} \tag{3.22}$$

for $l = 0, \ldots, k_{\min}$, where the familiar character identity (A.19) [5, 7] was used. The second line enables us to identify $F_l(\tau,z)$ as a spectral flow orbit of a system consisting of the level-$k_{\min}$, $N = 2$ minimal model, two complex fermions and a noncompact $N = 2$ CFT with some appropriate U(1)-charge lattice. Therefore, we can write

$$\begin{aligned} Z &= \int \frac{d\tau d\overline{\tau}}{\mathrm{Im}\tau}(\mathrm{Im}\tau)^{-3}\left|\eta^{-4}(\tau)\right|^2(\mathrm{Im}\tau)^{-\frac{1}{2}}\left|\eta^{-2}(\tau)\right|^2\sum_{l,\tilde{l}}N_{l,\tilde{l}}\frac{F_l(\tau,0)(F_{\tilde{l}}(\tau,0))^*}{|\eta^3(\tau)|^2} \\ &= \int \frac{d\tau d\overline{\tau}}{(\mathrm{Im}\tau)^2}(\mathrm{Im}\tau)^{-\frac{5}{2}}\left|\eta^{-5}(\tau)\right|^2\sum_{l,\tilde{l}}N_{l,\tilde{l}}\frac{F_l(\tau,0)(F_{\tilde{l}}(\tau,0))^*}{|\eta^4(\tau)|^2}, \end{aligned} \tag{3.23}$$

which is clearly modular invariant. This was regarded as a supersymmetric partition function modeling a deformed ADE singularity (3.21) with a six-dimensional Minkowski space.

In the threefold case, the relevant building blocks are

$$\begin{aligned} F_{l,2r}(\tau,z) \equiv \frac{1}{4}\sum_{m\in\mathbf{Z}_{4(k_{\min}+2)}} &\Big((\vartheta_3(\tau,z))^2\mathrm{ch}^{\mathrm{NS}}_{l,m}(\tau,z) - (-1)^{r-\frac{m}{2}}(\vartheta_4(\tau,z))^2\mathrm{ch}^{\widetilde{\mathrm{NS}}}_{l,m}(\tau,z) \\ &- (\vartheta_2(\tau,z))^2\mathrm{ch}^{\mathrm{R}}_{l,m}(\tau,z) + (-1)^{r-\frac{m}{2}+\frac{1}{2}}(\tilde{\vartheta}_1(\tau,z))^2\mathrm{ch}^{\widetilde{\mathrm{R}}}_{l,m}(\tau,z)\Big) \\ &\cdot\Theta_{(k_{\min}+2)2r-(k_{\min}+4)m,2(k_{\min}+2)(k_{\min}+4)}\left(\tau,\tfrac{z}{k_{\min}+2}\right) \end{aligned} \tag{3.24}$$

for $r \in \mathbf{Z}_{k_{\min}+4} + \frac{l}{2}$. As we explain in appendix, this $F_{l,2r}(\tau,z)$ is derived from Jacobi's identity and satisfies

$$\frac{1}{4}\chi_l^{(k_{\min})}(\tau,0)(\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 - \tilde{\vartheta}_1^4)(\tau,z) = \sum_{r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}} F_{l,2r}(\tau,z)\Theta_{2r,k_{\min}+4}(\tau,0). \tag{3.25}$$

The modular properties of $F_{l,2r}(\tau,z)$ can be read off from this equation. We can similarly write a modular invariant [13][7]

$$Z=\int\frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2}(\mathrm{Im}\tau)^{-\frac{3}{2}}\left|\eta(\tau)\right|^{-6}\sum_{l,\tilde{l}}\sum_{r\in\mathbf{Z}_{k_{\mathrm{min}}+4}+\frac{l}{2}}N_{l,\tilde{l}}\frac{F_{l,2r}(\tau,0)(F_{\tilde{l},2r}(\tau,0))^*}{\left|\eta^3(\tau)\right|^2},\qquad(3.26)$$

which can be regarded as the type II partition function for the ADE generalization of the conifold

$$W_{X_n}(z_1,z_2,z_3)+z_4^2=0,\qquad(3.27)$$

which is deformed to

$$W_{X_n}(z_1,z_2,z_3)+z_4^2=\mu z_0^{-\frac{2h(X_n)}{h(X_n)+2}}.\qquad(3.28)$$

Since the level of the $N=2$ minimal model is $k_{\mathrm{min}}=h(X_n)-2$, the level of the noncompact $N=2$ CFT is

$$\begin{aligned}\kappa&=\frac{2h(X_n)}{h(X_n)+2}+2\\&=\frac{2(k_{\mathrm{min}}+2)}{k_{\mathrm{min}}+4}+2.\end{aligned}\qquad(3.29)$$

For the $E_8\times E_8$ and SO(32) heterotic string theories, modular invariant partition functions on $\mathbf{R}^{d-1,1}$ ($d=6,4,2$) with only the continuous contributions have also been constructed in the second reference of [11]. (The $d=0$ case had been considered in [60] prior to that.)

3.3 Absence of localized modes

The spectrum of the $c=9$, $N=2$ representations used in the partition function (3.12) is shown in figure 3. What is nontrivial here is that [8] the level-1 U(1) theta functions determined by modular invariance and supersymmetry are consistent with unitarity. That is, the envelope of the lowest ends of the continuous spectra, which are set by the level-1 theta functions, *coincides exactly* with the lowest L_0 bound of possible $N=2$ representations corresponding to the continuous series of SL$(2,\mathbf{R})$.

By construction, there are only the representations coming from the continuous series of SL$(2,\mathbf{R})$. The graviton is massive due to the Liouville energy [61]. All the modes have continuous Liouville momenta and propagate into the extra dimension (that is, the Liouville direction). There are no localized massless modes.[8] This is also the case for the partition functions for the ADE singularity obtained in [13, 11]; they do not reflect the geometry in their massless spectrum [10].

[7] For simplicity, we take the coefficients of the modular invariant theta system $M^{k_{\mathrm{min}}}_{r,r'}$ [43] to be diagonal.

[8] There is a subtlety associated with the gapless spectrum. The appearance of this is the common feature of the spectrum for even k_{min}, the level of the $N=2$ minimal model coupled in the generalized models. See section 5.6.

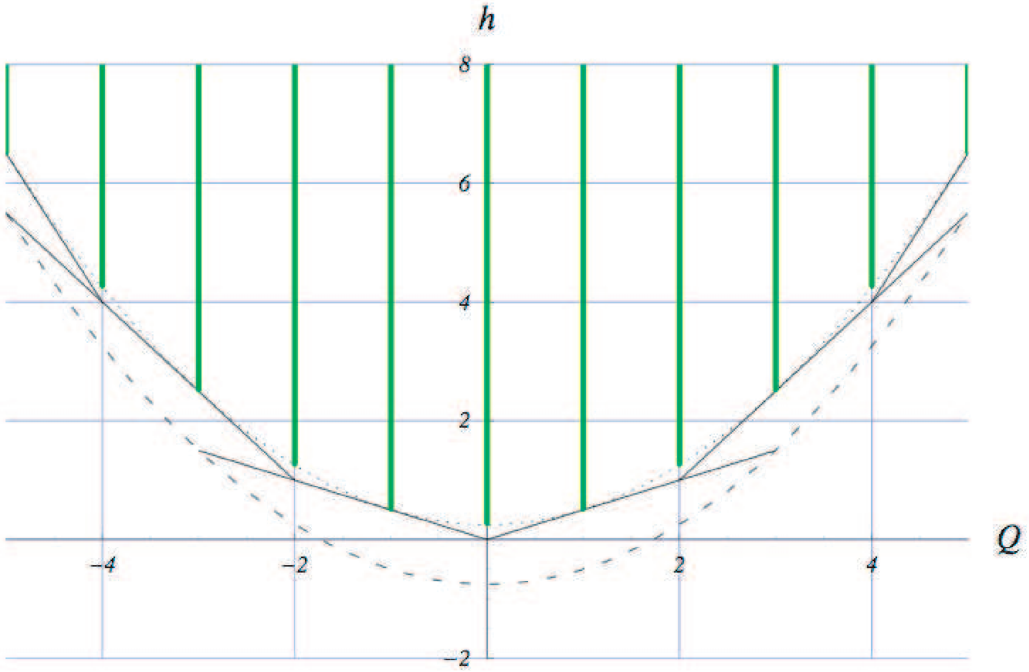

Figure 3: The spectrum of the $c = 9$, $N = 2$ representations in the old conifold partition function (NS sector). The green lines at odd (even) Q are the spectrum of representations contributing to Λ_1 (Λ_2).

In [13], new modular invariant partition functions including contributions from both the continuous and discrete series representations have been constructed for noncompact Calabi-Yau manifolds with an isolated singularity. (See also [62] for more recent related works.) They obtained them via the path-integral approach. They used the character decomposition technique developed in different but similar models [14, 15] to show the existence of the localized modes. In particular, they found [13] the correct chiral ring structures expected from the geometry of the ALE manifolds.

In the next section, based on this result, we construct spacetime supersymmetric partition functions (that is, the ones in which the fermions for the four-dimensional Minkowski space are coupled and GSO-projected before the continuous and discrete representations are separated) on the conifold-type threefolds for type II strings, and also for heterotic strings.

4. Partition functions of superstrings on noncompact singular Calabi-Yau threefolds

We start with the toroidal partition function of the $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ Kazama-Suzuki model [13]

$$Z_{\mathrm{CY}}^{(\mathrm{NS})}(\tau) = C\int_0^1 ds_1 \int_0^1 ds_2 \frac{|\vartheta_3(\tau, s_1\tau - s_2)|^2}{|\vartheta_1(\tau, s_1\tau - s_2)|^2} \sum_{v,w\in\mathbf{Z}} e^{-\frac{k\pi}{\tau_2}|(w+s_1)\tau-(v+s_2)|^2}. \qquad (4.1)$$

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This expression was obtained by a path integration [14, 15, 63–65] in the $H_3^+/\mathbf{R}$ gauged WZW model coupled to fermions. The detail of the derivation of (4.1) can be found in appendix C of ref. [13]. The partition functions for other spin structures $Z_{\rm CY}^{(\widetilde{\rm NS})}(\tau)$, $Z_{\rm CY}^{(\rm R)}(\tau)$ and $Z_{\rm CY}^{(\widetilde{\rm R})}(\tau)$ are given by similar expressions with ϑ_3 replaced by ϑ_4, ϑ_2 and ϑ_1, respectively. The overall constant C depends on the definition of the path integral measure and is arbitrary at this point, but later, after the discrete series contributions are separated, it is chosen to be $4k$ so that the discrete states partition function becomes a polynomial of q with integer coefficients. We should note that (4.1) is a formal expression because the s_1- and s_2-integrations diverge near $s_1 = s_2 = 0$, and hence need a regularization when we discuss the spectrum later.

By a Poisson resummation we may write

$$\begin{aligned}\sum_{v,w\in\mathbf{Z}} e^{-\frac{k\pi}{\tau_2}|(w+s_1)\tau-(v+s_2)|^2} &= \sum_{n,w\in\mathbf{Z}} e^{-\pi\tau_2\left(\frac{n^2}{k}+k(s_1+w)^2\right)-2\pi i n((s_1+w)\tau_1-s_2)} \\ &= \sqrt{\frac{\tau_2}{k}}\sum_{m,\tilde{m}} e^{-k\pi\tau_2 s_1^2} q^{\frac{m^2}{k}} e^{-2\pi i m(s_1\tau-s_2)}\bar{q}^{\frac{\tilde{m}^2}{k}} e^{+2\pi i\tilde{m}(s_1\bar{\tau}-s_2)}, \quad (4.2)\end{aligned}$$

where $m = \frac{n-kw}{2}$, $\tilde{m} = -\frac{n+kw}{2}$. They run over an appropriate direct sum of orthogonal lattices determined by $n, w \in \mathbf{Z}$.

$$k = \frac{2(k_{\min}+2)}{k_{\min}+4} \quad (k_{\min} = 0, 1, 2, \ldots). \tag{4.3}$$

4.1 $k_{\min} = 0$: the conifold

To get insight into how the GSO projection is accomplished before separating the continuous and discrete series representations, we consider the $k_{\min} = 0$ ($k = 1$) case first.

If $k = 1$, (4.2) becomes

$$\begin{aligned}(4.2) &= \sqrt{\tau_2} \sum_{\substack{m,\tilde{m}\in\mathbf{Z} \\ m=\tilde{m} \bmod 2}} e^{-\pi\tau_2 s_1^2} q^{m^2} e^{-2\pi i m(s_1\tau-s_2)}\bar{q}^{\tilde{m}^2} e^{+2\pi i\tilde{m}(s_1\bar{\tau}-s_2)} & (4.4) \\ &= \sqrt{\tau_2}\sum_{\nu\in\mathbf{Z_2}} e^{-\pi\tau_2 s_1^2}\Theta_{\nu,1}(\tau, s_2-s_1\tau)\left(\Theta_{\nu,1}(\tau, s_2-s_1\tau)\right)^* . & (4.5)\end{aligned}$$

The level-1 theta functions are precisely the ones which are used to construct a modular invariant partition function on the conifold consisting of only continuous series representations. This leads us to define, generalizing the continuous series result, the new functions

$$\hat{\Lambda}_1(\tau,z) \equiv \Theta_{1,1}(\tau,z)\Big(\vartheta_3(\tau,z)\vartheta_3(\tau,0)+\vartheta_4(\tau,z)\vartheta_4(\tau,0)\Big)-\Theta_{0,1}(\tau,z)\ \vartheta_2(\tau,z)\vartheta_2(\tau,0), \tag{4.6}$$

$$\hat{\Lambda}_2(\tau,z) \equiv \Theta_{0,1}(\tau,z)\Big(\vartheta_3(\tau,z)\vartheta_3(\tau,0)-\vartheta_4(\tau,z)\vartheta_4(\tau,0)\Big)-\Theta_{1,1}(\tau,z)\ \vartheta_2(\tau,z)\vartheta_2(\tau,0) \tag{4.7}$$

and write

$$Z_{\mathcal{M}_4\times\text{conifold}}(\tau) = C\int_0^1 ds_1\int_0^1 ds_2\sqrt{\tau_2}(q\bar{q})^{\frac{s_1^2}{4}}\frac{\left|\hat{\Lambda}_1(\tau,s_1\tau-s_2)\right|^2+\left|\hat{\Lambda}_2(\tau,s_1\tau-s_2)\right|^2}{|\eta(\tau)|^2|\vartheta_1(\tau,s_1\tau-s_2)|^2}, \tag{4.8}$$

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where $\tau = \tau_1 + i\tau_2$.

By definition of $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$, we see that $Z_{\mathcal{M}_4\times \mathrm{conifold}}(\tau)$ is a partition function for the $N = 2$ CFT for the conifold coupled to a complex fermion for the transverse space, with a GSO projection performed before the discrete series representations are separated.

Including the four-dimensional boson contributions, we obtain the full modular invariant partition function of type II strings on the conifold

$$Z^{\mathrm{full}}_{\mathcal{M}_4\times \mathrm{conifold}} = \int \frac{d\tau d\bar{\tau}}{\tau_2} \frac{1}{\tau_2^2 |\eta^2(\tau)|^2} Z_{\mathcal{M}_4\times \mathrm{conifold}}(\tau). \tag{4.9}$$

In the following we will show that $Z^{\mathrm{full}}_{\mathcal{M}_4\times \mathrm{conifold}}$:

(i) is modular invariant.

(ii) reduces to (3.12) if, after a certain regularization, divided by a divergent volume factor.

(iii) also contains contributions from the discrete series of $\mathrm{SL}(2,\mathbf{R})$, which transform as four-dimensional $\mathcal{N} = 2$ hyper/vector multiplets in type IIA/IIB string compactifications.

Here we note that, in going from (4.5) to (4.8), we have extended the summation region of (n, w) from $(\mathbf{Z}, \mathbf{Z})$ to $(\mathbf{Z}, \mathbf{Z}) \oplus (\mathbf{Z}+\frac{1}{2}, \mathbf{Z}+\frac{1}{2})$ to have a supersymmetric partition function. This is because we need $\Theta_{\nu,1}(\tau, s_2 - s_1\tau)(\Theta_{\tilde{\nu},1}(\tau, s_2 - s_1\tau))^*$ with $\nu\tilde{\nu} =$ odd in order to contain spacetime fermions. Therefore, we assume that n and w are allowed to take values in $\mathbf{Z} + \frac{1}{2}$ as well as in $\mathbf{Z}$. We also note that the particular z-dependence of the functions $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ is crucial to the construction, and is different from (3.7), (3.8) which are obtained by a composition of level-2 theta functions in Jacobi's quartic identity. (See appendix.)

4.2 Modular invariance of the type II conifold partition function

We first prove the modular S-invariance of (4.8). The following modular S-properties are well-known:

$$\tau_2 \to \frac{\tau_2}{|\tau|^2}, \tag{4.10}$$

$$\eta(\tau) \to \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\ \eta(\tau). \tag{4.11}$$

Also it is easy to see that

$$(q\bar{q})^{\frac{s_1^2}{4}} \to (q\bar{q})^{\frac{(1-s_2)^2}{4}} \left| e^{-\frac{\pi i}{2}\left(\tau(1-s_2)^2 + \frac{s_1^2}{\tau}\right)} \right|^2. \tag{4.12}$$

On the other hand, we have the following relation in general:

$$\Theta_{M,K}\left(\tau, \frac{s_1\tau - s_2}{-a}\right) \overset{\tau\to -\frac{1}{\tau}}{\longrightarrow} e^{\frac{Ki\pi}{2a^2}\left(\frac{s_1^2}{\tau} + \tau(1-s_2)^2 + 2(s_1 s_2 - s_1)\right)} \tag{4.13}$$
$$\cdot \sqrt{\frac{\tau}{2Ki}} \sum_{M' \in \mathbf{Z}_{2K}} e^{-\frac{MM'}{K}\pi i} \Theta_{M'+\frac{K}{a},K}\left(\tau, \frac{(1-s_2)\tau - s_1}{-a}\right)$$

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for any divisor a of a positive integer K. Comparing (4.13) with

$$\Theta_{M,K}(\tau,0) \overset{\tau\to-\frac{1}{\tau}}{\longrightarrow} \sqrt{\frac{\tau}{2Ki}} \sum_{M'\in\mathbf{Z}_{2K}} e^{-\frac{MM'}{K}\pi i}\Theta_{M',K}(\tau,0)\,, \tag{4.14}$$

we see that $\Theta_{M,K}\left(\tau, \frac{s_1\tau-s_2}{-a}\right)$ undergoes the following additional changes:

- The exponential factor.
- The replacement $(s_1,s_2)\to(1-s_2,s_1)$.
- The shift in the first subscript of the theta function ("spectral flow").

Using (4.13), we find

$$\vartheta_1(\tau,s_1\tau-s_2) \to -e^{\pi i\left(\frac{s_1^2}{\tau}+\tau(1-s_2)^2+2(s_1s_2-s_1)\right)}\vartheta_1(\tau,(1-s_2)\tau-s_1)\,, \tag{4.15}$$

$$\Theta_{m,1}(\tau,s_1\tau-s_2) \to e^{\frac{1}{2}\pi i\left(\frac{s_1^2}{\tau}+\tau(1-s_2)^2+2(s_1s_2-s_1)\right)} \sum_{m'\in\mathbf{Z}_2} e^{-mm'\pi i}\Theta_{m'-1,1}(\tau,(1-s_2)\tau-s_1)\,, \tag{4.16}$$

$$\Theta_{s,2}(\tau,s_1\tau-s_2) \to e^{\pi i\left(\frac{s_1^2}{\tau}+\tau(1-s_2)^2+2(s_1s_2-s_1)\right)} \sum_{s'\in\mathbf{Z}_4} e^{-\frac{ss'}{4}\pi i}\Theta_{s-2,2}(\tau,(1-s_2)\tau-s_1)\,. \tag{4.17}$$

Since $2\pi i(s_1s_2-s_1)$ is pure imaginary, it is irrelevant if the absolute value is taken. Then the exponential factors of $e^{\mathrm{const.}\times\pi i\left(\frac{s_1^2}{\tau}+\tau(1-s_2)^2\right)}$ arising from various factors of (4.8) cancel out. The replacement $(s_1,s_2)\to(1-s_2,s_1)$ acts trivially on $\int_0^1 s_1\int_0^1 s_2$. Therefore, since

$$\frac{|\Lambda_1(\tau,0)|^2+|\Lambda_2(\tau,0)|^2}{|\eta^3(\tau)|^2} \tag{4.18}$$

is modular invariant, we have only to worry about the shift of m' and s' in the theta functions in $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$. It turns out that they simply amount to the permutation

$$\hat{\Lambda}_1 \to \hat{\Lambda}_2, \tag{4.19}$$

$$\hat{\Lambda}_2 \to \hat{\Lambda}_1, \tag{4.20}$$

which obviously preserves (4.8). Thus we have proved that $Z_{\mathcal{M}_4\times\mathrm{conifold}}(\tau)$ is modular S-invariant.

The proof of the modular T-invariance is easier. Since

$$\hat{\Lambda}_1(\tau+1,z) = i\hat{\Lambda}_1(\tau,z), \tag{4.21}$$

$$\hat{\Lambda}_1(\tau+1,z) = -\hat{\Lambda}_2(\tau,z) \tag{4.22}$$

hold independently of z, all we need to do is to examine the effect of the change of $s_1\tau-s_2$, which amounts to a change of variables $(s_1,s_2)\to(s_1,s_2-s_1)$. In fact, the integrand of

$Z_{\mathcal{M}_4\times\mathrm{conifold}}(\tau)$ (4.8) is periodic (with a period of 1) in s_2, so the integral is invariant under the change of variables. Thus $Z_{\mathcal{M}_4\times\mathrm{conifold}}(\tau)$ is also T-invariant.

Again, we note that this proof for the modular invariance is a formal one because $Z_{\mathcal{M}_4\times\mathrm{conifold}}(\tau)$ (as well as $Z_{\mathcal{M}_4\times CY(X_n)}(\tau)$ in the next section) is a divergent quantity. In section 5.3, we introduce a regularization which is not modular invariant. The situation is reminiscent of anomalies in gauge theory; when the continuous and discrete contributions are separated after the regularization, the continuous piece forms a modular invariant while the discrete one does not close under modular transformations. The number of massless discrete states should not depend on how the partition function is regularized since it reflects the moduli of the noncompact Calabi-Yau (or the NS5-branes), as we see in the subsequent sections.

4.3 Type II string partition functions for general k

In this section, we extend the discussion for the conifold to more general singularities in which a nontrivial $N=2$ minimal model with a general non-negative integer level k_{min} is coupled to the noncompact $N=2$ coset theory.

The expression of the new partition function for the conifold $Z_{\mathcal{M}_4\times\mathrm{conifold}}(\tau)$ (4.8) is very suggestive; it is similar in form to the old partition function (3.11). In particular, the alternating sum is realized in similar functions $\Lambda_i(\tau,z)$ $(i=1,2)$ and $\hat{\Lambda}_i(\tau,z)$ $(i=1,2)$, which differ only in the z-dependences. This motivates us to define

$$\begin{aligned}\hat{F}_{l,2r}(\tau,z) = \frac{1}{4}\sum_{m\in\mathbf{Z}_{4(k_{\mathrm{min}}+2)}} &\Big(\vartheta_3(\tau,0)\vartheta_3(\tau,z)\mathrm{ch}^{\mathrm{NS}}_{l,m}(\tau,0) - (-1)^{r-\frac{m}{2}}\vartheta_4(\tau,0)\vartheta_4(\tau,z)\mathrm{ch}^{\widetilde{\mathrm{NS}}}_{l,m}(\tau,0)\\ &\qquad - \vartheta_2(\tau,0)\vartheta_2(\tau,z)\mathrm{ch}^{\mathrm{R}}_{l,m}(\tau,0)\Big)\\ &\qquad \cdot\Theta_{(k_{\mathrm{min}}+2)2r-(k_{\mathrm{min}}+4)m,2(k_{\mathrm{min}}+2)(k_{\mathrm{min}}+4)}\left(\tau,\tfrac{z}{k_{\mathrm{min}}+4}\right).\end{aligned} \tag{4.23}$$

Again, this $\hat{F}_{l,2r}(\tau,z)$ is obtained from $F_{l,2r}(\tau,z)$ (3.24), which was defined in [9] to construct the modular invariant partition function for the ADE type conifold-like singularities with only the continuous series representations. Remarkably, the level-$2(k_{\mathrm{min}}+2)(k_{\mathrm{min}}+4)$ theta functions are precisely the ones which appear in the U(1)-charge lattice decomposition

$$\begin{aligned}(4.2) &= \sqrt{\frac{\tau_2}{k}}\sum_{j_1,j_2\in\mathbf{Z}}\ \sum_{m\in\mathbf{Z}_{4(k_{\mathrm{min}}+2)}}\ \sum_{r\in\mathbf{Z}_{2(k_{\mathrm{min}}+4)}} e^{-k\pi\tau_2 s_1^2}\\ &\quad\cdot q^{\frac{k_{\mathrm{min}}+4}{2(k_{\mathrm{min}}+2)}\left(2(k_{\mathrm{min}}+2)(j_1-j_2)+\frac{m}{2}-\frac{(k_{\mathrm{min}}+2)r}{k_{\mathrm{min}}+4}\right)^2} e^{-2\pi i(s_1\tau-s_2)\left(2(k_{\mathrm{min}}+2)(j_1-j_2)+\frac{m}{2}-\frac{(k_{\mathrm{min}}+2)r}{k_{\mathrm{min}}+4}\right)}\\ &\quad\cdot \bar{q}^{\frac{k_{\mathrm{min}}+4}{2(k_{\mathrm{min}}+2)}\left(2(k_{\mathrm{min}}+2)(-j_1-j_2)-\frac{m}{2}-\frac{(k_{\mathrm{min}}+2)r}{k_{\mathrm{min}}+4}\right)^2} e^{+2\pi i(s_1\bar{\tau}-s_2)\left(2(k_{\mathrm{min}}+2)(-j_1-j_2)-\frac{m}{2}-\frac{(k_{\mathrm{min}}+2)r}{k_{\mathrm{min}}+4}\right)}\end{aligned} \tag{4.24}$$

$$\begin{aligned}&= \sqrt{\frac{\tau_2}{k}}\sum_{m\in\mathbf{Z}_{4(k_{\mathrm{min}}+2)}}\ \sum_{r\in\mathbf{Z}_{2(k_{\mathrm{min}}+4)}} e^{-k\pi\tau_2 s_1^2}\frac{1}{2}\\ &\quad\cdot\Theta_{(k_{\mathrm{min}}+4)m-2(k_{\mathrm{min}}+2)r,2(k_{\mathrm{min}}+2)(k_{\mathrm{min}}+4)}\left(\tau,\frac{s_2-s_1\tau}{k_{\mathrm{min}}+4}\right)\end{aligned}$$

$$\cdot\left(\Theta_{-(k_{\min}+4)m-2(k_{\min}+2)r,2(k_{\min}+2)(k_{\min}+4)}\left(\tau,\frac{s_2-s_1\tau}{k_{\min}+4}\right)\right)^*, \tag{4.25}$$

where $*$ denotes the complex conjugate. Therefore, $\hat{F}_{l,2r}(\tau,z)$ specifies a particular way of GSO projection in the $N=2$ minimal model, the transverse fermion and the noncompact $N=2$ coset Hilbert spaces. Using $\hat{F}_{l,2r}(\tau,z)$, we can similarly write a modular invariant expression

$$\begin{aligned} Z_{\mathcal{M}_4\times CY(X_n)}(\tau) &= C\int_0^1 ds_1\int_0^1 ds_2\sqrt{\frac{\tau_2}{k}}(q\overline{q})^{\frac{ks_1^2}{4}} \\ &\cdot\sum_{l,\tilde{l}} N_{l,\tilde{l}}\sum_{r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}}\frac{\hat{F}_{l,2r}(\tau,s_1\tau-s_2)\left(\hat{F}_{\tilde{l},2r}(\tau,s_1\tau-s_2)\right)^*}{|\eta(\tau)|^2|\vartheta_1(\tau,s_1\tau-s_2)|^2} \end{aligned} \tag{4.26}$$

for general $k_{\min}$. $N_{l,\tilde{l}}$ is the coefficients of the X_n ($X=A,D$, or E) modular invariant [66, 67]. Since $N_{l,\tilde{l}}$ vanishes if $l-\tilde{l}=1$ (mod 2) for any modular invariant, the summation over $r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}$ is equivalent to the one over $r\in\mathbf{Z}_{k_{\min}+4}+\frac{\tilde{l}}{2}$. If $k_{\min}=0$, $Z_{\mathcal{M}_4\times CY(A_1)}(\tau)$ is reduced to $Z_{\mathcal{M}_4\times \mathrm{conifold}}(\tau)$ (4.8).

The proof of the modular invariance of $Z_{\mathcal{M}_4\times CY(X_n)}(\tau)$ is parallel to the conifold case. Again, the only nontrivial point is the τ-dependence through the z-argument. In the present case the modular S-transformation simply permutes $\hat{F}$'s cyclically, and $Z_{\mathcal{M}_4\times CY(X_n)}(\tau)$ as a whole remains invariant. The proof of the modular T-invariance is also similar.

4.4 Heterotic string partition functions for general k

Once we have a modular invariant partition function for type II strings, we can easily convert it to one for heterotic strings by a standard procedure [5], as we review in appendix. All we need to do is replace the holomorphic $\hat{F}_{l,2r}(\tau,s_1\tau-s_2)$ in (4.26) with $\hat{F}^{E_8\times E_8}_{l,2r}(\tau,s_1\tau-s_2)/\eta^{12}(\tau)$ (C.23)–(C.27) for the $E_8\times E_8$ theory, and with $\hat{F}^{\mathrm{SO}(32)}(\tau,s_1\tau-s_2)/\eta^{12}(\tau)$ (C.28)–(C.32) for the SO(32) theory. The anti-holomorphic $(\hat{F}_{\tilde{l},2r}(\tau,s_1\tau-s_2))^*$ is left unchanged. Since $\hat{F}^{E_8\times E_8}_{l,2r}(\tau,s_1\tau-s_2)/\eta^{12}(\tau)$ or $\hat{F}^{\mathrm{SO}(32)}(\tau,s_1\tau-s_2)/\eta^{12}(\tau)$ transforms in the same way as $\hat{F}_{l,2r}(\tau,s_1\tau-s_2)$ does, the resulting heterotic partition function is automatically modular invariant. Their massless spectra will be investigated in the next section.

5. Separation of the discrete series contributions

We now separate the contributions from the discrete series representations from the partition functions obtained in the previous section. In section 5.1, we first define modules of various algebras and describe relevant spectral flow operations in them, which are needed later. Then we consider the separation for type II strings from section 5.2 through section 5.6, and for heterotic strings in section 5.7.

5.1 Modules and spectral flows

- *The* SL*(2,R) Kac-Moody algebra module* $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{\pm,(h,l_0)}$.

The affine $\mathrm{SL}(2,\mathbf{R})$ algebra relations are [42]

$$[J^3_n,\ \ J^3_m] = -\frac{\kappa}{2}n\delta_{n,-m}, \tag{5.1}$$

$$[J^3_n,\ \ J^{\pm}_m] = \pm J^{\pm}_{n+m}, \tag{5.2}$$

$$[J^+_n,\ \ J^-_m] = -\kappa n\delta_{n,-m} + 2J^3_{n+m} \tag{5.3}$$

for $n,\, m \in \mathbf{Z}$, where

$$\kappa = k+2. \tag{5.4}$$

The Virasoro generators are

$$\begin{aligned} L_0^{\mathrm{SL}(2,\mathbf{R})} &= \frac{1}{2(\kappa-2)}\Bigg(J^+_0J^-_0 + J^-_0J^+_0 - 2(J^3_0)^2 \\ &\qquad\qquad +2\sum_{m=1}^{\infty}\left(J^+_{-m}J^-_m + J^-_{-m}J^+_m - 2J^3_{-m}J^3_m\right)\Bigg), \\ L_n^{\mathrm{SL}(2,\mathbf{R})} &= \frac{1}{2(\kappa-2)}\sum_{-\infty}^{\infty}\left(J^+_{-m}J^-_m + J^-_{-m}J^+_m - 2J^3_{-m}J^3_m\right). \end{aligned} \tag{5.5}$$

We define $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{\pm,(h,l_0)}$ as an $\mathrm{SL}(2,\mathbf{R})$ Kac-Moody algebra module generated from a state $|h,l_0\rangle$ such that

$$L_0^{\mathrm{SL}(2,\mathbf{R})}|h,l_0\rangle = h|h,l_0\rangle, \tag{5.6}$$

$$J^3_0|h,l_0\rangle = l_0|h,l_0\rangle, \tag{5.7}$$

$$L_n^{\mathrm{SL}(2,\mathbf{R})}|h,l_0\rangle = J^3_n|h,l_0\rangle = J^+_n|h,l_0\rangle = J^-_n|h,l_0\rangle = 0, \quad (n>0) \tag{5.8}$$

$$J^{\mp}_0|h,l_0\rangle = 0. \tag{5.9}$$

The character for a *generic* representation is given by

$$\mathrm{Tr}_{\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{\pm,(h,l_0)}} q^{L_0^{\mathrm{SL}(2,\mathbf{R})}} y^{J^3_0} = \frac{\pm i q^{\frac{1}{8}+h} y^{\mp\frac{1}{2}+l_0}}{\vartheta_1(\tau,z)}. \tag{5.10}$$

Let us define the spectral-flow operation

$$J^{\pm}_n = \tilde{J}^{\pm}_{n\mp w}, \tag{5.11}$$

$$J^3_n = \tilde{J}^3_n + \frac{\kappa w}{2}\delta_{n,0}, \tag{5.12}$$

$$L_n^{\mathrm{SL}(2,\mathbf{R})} = \tilde{L}_n^{\mathrm{SL}(2,R)} - w\tilde{J}^3_n - \frac{\kappa w^2}{4}\delta_{n,0}, \tag{5.13}$$

then the tilde generators also satisfy the same algebra relations as those without tildes, so it is an isomorphism. In particular, if we set $w=1$, then

$$J^-_0 = \tilde{J}^-_1, \tag{5.14}$$

$$J^+_1 = \tilde{J}^+_0, \tag{5.15}$$

and $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(h,l_0)}$ as a module generated by J_n's can be identified to be $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(h+l_0-\frac{\kappa}{4},l_0-\frac{\kappa}{2})}$ as a module generated by $\tilde{J}_n$'s. Therefore, for any function $f(L_0^{\mathrm{SL}(2,\mathbf{R})}, J_0^3)$, the following equation holds true:

$$\mathrm{Tr}_{\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(h,l_0)}} f\left(L_0^{\mathrm{SL}(2,\mathbf{R})}, J_0^3\right) = \mathrm{Tr}_{\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(h+l_0-\frac{\kappa}{4},l_0-\frac{\kappa}{2})}} f\left(\tilde{L}_0^{\mathrm{SL}(2,R)} - \tilde{J}_0^3 - \frac{\kappa}{4},\ \tilde{J}_0^3 + \frac{\kappa}{2}\right). \quad (5.16)$$

- *The free fermion module $\mathcal{H}_{\nu,2}$ ($\nu \in \mathbf{Z}_4$).*

  The complex fermion algebra is generated by $\psi_r^{\pm}$, where $r \in \mathbf{Z} + \frac{1}{2}$ in the NS sector and $r \in \mathbf{Z}$ in the Ramond sector, with the relations

$$\{\psi_r^+,\ \psi_s^+\} = \{\psi_r^-,\ \psi_s^-\} = 0, \quad (5.17)$$
$$\{\psi_r^+,\ \psi_s^-\} = \{\psi_r^-,\ \psi_s^+\} = \delta_{r,-s}. \quad (5.18)$$

The L_0 and fermion number operators are

$$L_0^{(\mathrm{NS})} = \sum_{r\in\mathbf{Z}+\frac{1}{2},>0} \frac{r}{2}\left(\psi_{-r}^+\psi_r^- + \psi_{-r}^-\psi_r^+\right), \quad (5.19)$$
$$L_0^{(\mathrm{R})} = \sum_{r\in\mathbf{Z},>0} \frac{r}{2}\left(\psi_{-r}^+\psi_r^- + \psi_{-r}^-\psi_r^+\right) + \frac{1}{8}, \quad (5.20)$$
$$F^{(\mathrm{NS})} = \frac{1}{2}\sum_{r\in\mathbf{Z}+\frac{1}{2},>0} \left(\psi_{-r}^+\psi_r^- - \psi_{-r}^-\psi_r^+\right), \quad (5.21)$$
$$F^{(\mathrm{R})} = \frac{1}{2}\psi_0^+\psi_0^- + \frac{1}{2}\sum_{r\in\mathbf{Z},>0} \left(\psi_{-r}^+\psi_r^- - \psi_{-r}^-\psi_r^+\right) - \frac{1}{2}. \quad (5.22)$$

As usual, we introduce the NS ground state $|0\rangle_{\mathrm{NS}}$ such that

$$\psi_r^{\pm}|0\rangle_{\mathrm{NS}} = 0 \quad (5.23)$$

for $r = \frac{1}{2}, \frac{3}{2}, \ldots$, and the Ramond ground state $|0\rangle_{\mathrm{R}}$ such that

$$\psi_r^{+}|0\rangle_{\mathrm{R}} = 0 \quad (5.24)$$

for $r = 1, 2, \ldots$, while

$$\psi_r^{-}|0\rangle_{\mathrm{R}} = 0 \quad (5.25)$$

for $r = 0, 1, 2, \ldots$. Then

$$L_0^{(NS)}|0\rangle_{\mathrm{NS}} = 0, \quad (5.26)$$
$$F^{(NS)}|0\rangle_{\mathrm{NS}} = 0, \quad (5.27)$$
$$L_0^{(R)}|0\rangle_{\mathrm{R}} = \frac{1}{8}|0\rangle_{\mathrm{R}}, \quad (5.28)$$
$$F^{(R)}|0\rangle_{\mathrm{R}} = -\frac{1}{2}|0\rangle_{\mathrm{R}}. \quad (5.29)$$

Let us call the free fermion modules generated from these ground states $\mathcal{H}^{(NS)}$ and $\mathcal{H}^{(R)}$, respectively. We also define $\mathcal{H}_{\nu,2}$ ($\nu \in \mathbf{Z}_4$) to be a free fermion module such that

$$\mathrm{Tr}_{\mathcal{H}_{\nu,2}} q^{L_0^{(\nu)}} y^{F^{(\nu)}} = q^{\frac{1}{24}} \frac{\Theta_{\nu,2}(\tau,z)}{\eta(\tau)}, \tag{5.30}$$

where

$$\begin{aligned} L_0^{(\nu)} &= L_0^{(\mathrm{NS})} \quad \text{if } \nu = 0,2, & (5.31)\\ &= L_0^{(\mathrm{R})} \quad \text{if } \nu = \pm 1, & (5.32)\\ F^{(\nu)} &= F^{(\mathrm{NS})} \quad \text{if } \nu = 0,2, & (5.33)\\ &= F^{(\mathrm{R})} \quad \text{if } \nu = \pm 1. & (5.34)\end{aligned}$$

Clearly, $\mathcal{H}_{0,2}$ ($\mathcal{H}_{2,2}$) consists of even (odd) $F^{(\mathrm{NS})}$ states in $\mathcal{H}^{(NS)}$, and similarly $\mathcal{H}_{1,2}$ ($\mathcal{H}_{-1,2}$) consists of states with $F^{(R)} = +\frac{1}{2}+$ even- (odd-) integer in $\mathcal{H}^{(\mathrm{R})}$. Also

$$\begin{aligned} \mathcal{H}^{(\mathrm{NS})} &= \mathcal{H}_{0,2} \oplus \mathcal{H}_{2,2}, & (5.35)\\ \mathcal{H}^{(\mathrm{R})} &= \mathcal{H}_{1,2} \oplus \mathcal{H}_{-1,2}. & (5.36)\end{aligned}$$

For both the NS and the Ramond sectors,

$$\psi_r^{\pm(\mathrm{NS,R})} = \tilde{\psi}_{r+1}^{\pm(\mathrm{NS,R})} \tag{5.37}$$

is an isomorphism and maps $\mathcal{H}^{(\mathrm{NS})}$ to $\mathcal{H}^{(\mathrm{NS})}$, and $\mathcal{H}^{(\mathrm{R})}$ to $\mathcal{H}^{(\mathrm{R})}$. We also have, any function $f(L_0^{(\nu)}, F^{(\nu)})$, the following equation

$$\mathrm{Tr}_{\mathcal{H}_{\nu,2}} f\left(L_0^{(\nu)}, F^{(\nu)}\right) = \mathrm{Tr}_{\mathcal{H}_{\nu+2,2}} f\left(\tilde{L}_0^{(\nu+2)} - \tilde{F}^{(\nu+2)} + \frac{1}{2},\ \tilde{F}^{(\nu+2)} - 1\right). \tag{5.38}$$

- *The free boson module $\mathcal{H}_{m,K}$ ($K = 2(k_{\mathrm{min}}+2)(k_{\mathrm{min}}+4)$, $m \in \mathbf{Z}_{2K}$).*

  Let us consider a free scalar field $\phi(z)$ with the OPE

$$\phi(z)\phi(w) \sim -\log(z-w) \tag{5.39}$$

with the energy-momentum tensor and the U(1) current

$$\begin{aligned} T^{\mathrm{U}(1)}(z) &= -\frac{1}{2}(\partial\phi(z))^2, & (5.40)\\ J^{\mathrm{U}(1)}(z) &= i\sqrt{\frac{K}{2}}\partial\phi(z). & (5.41)\end{aligned}$$

for some integer $K = 2(k_{\mathrm{min}}+2)(k_{\mathrm{min}}+4)$. Let $L_0^{\mathrm{U}(1)}$ and $J_0^{\mathrm{U}(1)}$ be their zeromodes. Then $\mathcal{H}_{m,K}$ ($m \in \mathbf{Z}_{2K}$) is defined to be a (reducible) free boson module such that

$$\mathrm{Tr}_{\mathcal{H}_{m,K}} q^{L_0^{\mathrm{U}(1)}} y^{J_0^{\mathrm{U}(1)}} = q^{\frac{1}{24}} \frac{\Theta_{m,K}(\tau,z)}{\eta(\tau)}, \tag{5.42}$$

and hence

$$\mathrm{Tr}_{\mathcal{H}_{m,K}} q^{L_0^{\mathrm{U}(1)}} y^{\frac{J_0^{\mathrm{U}(1)}}{k_{\mathrm{min}}+4}} = q^{\frac{1}{24}} \frac{\Theta_{m,K}(\tau, \frac{z}{k_{\mathrm{min}}+4})}{\eta(\tau)}. \tag{5.43}$$

$\mathcal{H}_{m,K}$ is a direct product of free boson modules generated from the ground states

$$e^{i\sqrt{2K}(n+\frac{m}{2K})\phi(0)}|0\rangle \quad (n \in \mathbf{Z}), \tag{5.44}$$

where $|0\rangle$ is the $J_0^{\mathrm{U}(1)} = 0$ ground state.

The replacement

$$\begin{aligned} L_0^{\mathrm{U}(1)} &= \tilde{L}_0^{\mathrm{U}(1)} - \frac{\tilde{J}_0^{\mathrm{U}(1)}}{k_{\mathrm{min}}+4} + \frac{k_{\mathrm{min}}+2}{2(k_{\mathrm{min}}+4)}, &(5.45)\\ J_0^{\mathrm{U}(1)} &= \tilde{J}_0^{\mathrm{U}(1)} - (k_{\mathrm{min}}+2) &(5.46)\end{aligned}$$

is a spectral flow by $2(k_{\mathrm{min}}+2)$ units, and hence is an isomorphism. As before, we have a relation

$$\begin{aligned} &\mathrm{Tr}_{\mathcal{H}_{m,K}} f\left(L_0^{\mathrm{U}(1)}, J_0^{\mathrm{U}(1)}\right) \\ &\quad = \mathrm{Tr}_{\mathcal{H}_{m+2(k_{\mathrm{min}}+2),K}} f\left(\tilde{L}_0^{\mathrm{U}(1)} - \frac{\tilde{J}_0^{\mathrm{U}(1)}}{k_{\mathrm{min}}+4} + \frac{k_{\mathrm{min}}+2}{2(k_{\mathrm{min}}+4)},\ \tilde{J}_0^{\mathrm{U}(1)} - (k_{\mathrm{min}}+2)\right) \end{aligned} \tag{5.47}$$

for any $f(L_0^{\mathrm{U}(1)}, J_0^{\mathrm{U}(1)})$.

- *The $N=2$ minimal superconformal algebra module $\mathcal{H}_m^{(k_{\mathrm{min}})l,s}$.*

  Finally, we define the $N=2$ minimal superconformal algebra module $\mathcal{H}_m^{(k_{\mathrm{min}})l,s}$ such that

$$\mathrm{Tr}_{\mathcal{H}_m^{(k_{\mathrm{min}})l,s}} q^{L_0^{N=2}} y^{J_0^{N=2}} = q^{\frac{c_{\mathrm{min}}}{24}} \chi_m^{(k_{\mathrm{min}})l,s}(\tau, z). \tag{5.48}$$

  In appendix we collect useful formulas of the $N=2$ minimal characters $\chi_m^{(k_{\mathrm{min}})l,s}(\tau,z) \equiv \chi_m^{l,s}(\tau,z)$, where the k_{min}-dependence is suppressed for notational simplicity. We do not need spectral flow formulas for them because they do not have the denominator U(1) charge of the gauged WZW model.

5.2 The flow-orbit representation of the partition functions

To extract the discrete series contributions from (4.26), we will write it as a trace of some operator over appropriate modules defined in the previous subsection. First, we note that the function $F_{l,2r}(\tau,z)$, introduced in [9] to construct modular invariants containing only the continuous series, can be written as a spectral flow orbit with respect to the $N=2$ U(1) charge (which is not the same as the denominator U(1) charge of the SL$(2,\mathbf{R})$/U(1) coset counted by J_0^{tot} below, as emphasized in [13]), as shown in appendix. Since $F_{l,2r}(\tau,z)$

and $\hat{F}_{l,2r}(\tau,z)$ differ only in the z-dependences of theta functions, $\hat{F}_{l,2r}(\tau,z)$ can also be expressed as a similar alternating sum (see (B.26))

$$\hat{F}_{l,2r}(\tau,z) = \frac{1}{2}\sum_{\nu\in\mathbf{Z}_{4(k_{\min}+2)}}\sum_{\substack{\nu_0,\nu_1,\nu_2\in Z_2\\ \nu_0+\nu_1+\nu_2\\ \equiv 1(\mathrm{mod}2)}}(-1)^{\nu}\chi_{l+\nu}^{l,l-2r+2\nu_0+\nu}(\tau,0)\Theta_{2\nu_1+\nu,2}(\tau,0)$$
$$\cdot\,\Theta_{2\nu_2+\nu,2}(\tau,z)\Theta_{(k_{\min}+2)2r-(k_{\min}+4)(l+\nu),2(k_{\min}+2)(k_{\min}+4)}\left(\tau,\frac{z}{k_{\min}+4}\right). \tag{5.49}$$

It motivates us to define [41]

$$\mathcal{H}^{(\nu)}_{F_{l,2r}} \equiv \bigoplus_{\substack{\nu_0,\nu_1,\nu_2\in Z_2\\ \nu_0+\nu_1+\nu_2\\ \equiv 1(\mathrm{mod}2)}}\left(\mathcal{H}^{(k_{\min})l,l-2r+2\nu_0+\nu}_{l+\nu}\otimes\mathcal{H}_{2\nu_1+\nu,2}\otimes\mathcal{H}_{2\nu_2+\nu,2}\right)$$
$$\otimes\,\mathcal{H}_{(k_{\min}+2)2r-(k_{\min}+4)(l+\nu),2(k_{\min}+2)(k_{\min}+4)}. \tag{5.50}$$

We can then write

$$\int_0^1 ds_1\int_0^1 ds_2\sqrt{\frac{\tau_2}{k}}(q\bar{q})^{\frac{ks_1^2}{4}}\frac{\hat{F}_{l,2r}(\tau,s_1\tau-s_2)\left(\hat{F}_{\tilde{l},2r}(\tau,s_1\tau-s_2)\right)^*}{|\eta(\tau)\vartheta_1(\tau,s_1\tau-s_2)|^2}$$
$$=\int_0^1 ds_1\int_0^1 ds_2\sqrt{\frac{\tau_2}{k}}(q\bar{q})^{\frac{ks_1^2}{4}}\frac{|\eta^2(\tau)|^2}{4}\sum_{\nu,\tilde{\nu}\in\mathbf{Z}_{4(k_{\min}+2)}}(-1)^{\nu+\tilde{\nu}}$$
$$\cdot\mathrm{Tr}_{\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)}\otimes\mathcal{H}^{(\nu)}_{F_{l,2r}}\right)\otimes\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)}\otimes\mathcal{H}^{(\tilde{\nu})}_{F_{\tilde{l},2r}}\right)}$$
$$\cdot\, q^{L_0^{\mathrm{SL}(2,\mathbf{R})}+L_0^{N=2}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{\mathrm{U}(1)}-\frac{1}{4}-\frac{c_{\min}}{24}+s_1\left(J_0^3+F^{(\nu)}+\frac{J_0^{\mathrm{U}(1)}}{k_{\min}+4}+\frac{1}{2}\right)}$$
$$\cdot\, \bar{q}^{\tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})}+\tilde{L}_0^{N=2}+\tilde{L}_0^{(\tilde{\nu})}+\tilde{L}_0^{(\tilde{\nu})}+\tilde{L}_0^{\mathrm{U}(1)}-\frac{1}{4}-\frac{c_{\min}}{24}+s_1\left(\tilde{J}_0^3+\tilde{F}^{(\tilde{\nu})}+\frac{\tilde{J}_0^{\mathrm{U}(1)}}{k_{\min}+4}+\frac{1}{2}\right)}$$
$$\cdot\, e^{-2\pi i s_2\left(J_0^3+F^{(\nu)}+\frac{J_0^{\mathrm{U}(1)}}{k_{\min}+4}-\tilde{J}_0^3-\tilde{F}^{(\tilde{\nu})}-\frac{\tilde{J}_0^{\mathrm{U}(1)}}{k_{\min}+4}\right)}. \tag{5.51}$$

The s_2 integral yields a constraint

$$(J_0^{\mathrm{tot}}\equiv)\ J_0^3+F^{(\nu)}+\frac{J_0^{\mathrm{U}(1)}}{k_{\min}+4} = \tilde{J}_0^3+\tilde{F}^{(\tilde{\nu})}+\frac{\tilde{J}_0^{\mathrm{U}(1)}}{k_{\min}+4}\ (\equiv\tilde{J}_0^{\mathrm{tot}}). \tag{5.52}$$

Using the Fourier transformation

$$\sqrt{\frac{\tau_2}{k}}(q\bar{q})^{\frac{ks_1^2}{4}} = \frac{1}{k}\int_{-\infty}^{\infty}dc\; e^{-\frac{\pi}{k\tau_2}c^2-2\pi i c s_1}, \tag{5.53}$$

we can perform the s_1 integral as

$$=\frac{|\eta^2(\tau)|^2}{4k}\int_{-\infty}^{\infty}dp\sum_{\nu,\tilde{\nu}\in\mathbf{Z}_{4(k_{\min}+2)}}(-1)^{\nu+\tilde{\nu}}\mathrm{Tr}_{\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)}\otimes\mathcal{H}^{(\nu)}_{F_{l,2r}}\right)\times\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)}\otimes\mathcal{H}^{(\tilde{\nu})}_{F_{\tilde{l},2\tilde{r}}}\right)}$$

$$\cdot q^{L_0^{\mathrm{SL}(2,\mathbf{R})}+L_0^{N=2}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{\mathrm{U}(1)}-\frac{1}{4}-\frac{c_{\min}}{24}}\bar{q}^{\tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})}+\tilde{L}_0^{N=2}+\tilde{L}_0^{(\tilde{\nu})}+\tilde{L}_0^{(\tilde{\nu})}+\tilde{L}_0^{\mathrm{U}(1)}-\frac{1}{4}-\frac{c_{\min}}{24}}$$
$$\cdot\frac{(q\bar{q})^{\frac{1}{k}\left(p+\frac{ik}{2}\right)^2+J_0^{\mathrm{tot}}+\frac{k+2}{4}}-(q\bar{q})^{\frac{p^2}{k}}}{-2\pi(ip+J_0^{\mathrm{tot}}+\frac{1}{2})}, \tag{5.54}$$

where we set $c = 2\tau_2 p$.

The first term of the numerator contains J_0^{tot} in its exponent. If we use the isomorphisms of various modules in the previous section, we can eliminate this J_0^{tot} dependence, and also the module over which the first trace is taken changes from $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)} \otimes \mathcal{H}^{(\nu)}_{F_{l,2r}}$ to $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(-\frac{\kappa}{4},-\frac{\kappa}{2})} \otimes \mathcal{H}^{(\nu)}_{F_{l,2(r+1)}}$ ($\kappa = k+2$), and similarly in the anti-holomorphic sector:

$$\begin{aligned}
(5.51) = \frac{|\eta^2(\tau)|^2}{4k} &\sum_{\nu,\tilde{\nu}\in\mathbf{Z}_{4(k_{\min}+2)}} (-1)^{\nu+\tilde{\nu}} \\
&\cdot\Bigg(\mathrm{Tr}_{\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(-\frac{\kappa}{4},-\frac{\kappa}{2})}\otimes\mathcal{H}^{(\nu)}_{F_{l,2(r+1)}}\right)\otimes\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(-\frac{\kappa}{4},-\frac{\kappa}{2})}\mathcal{H}^{(\tilde{\nu})}_{F_{l,2(r+1)}}\right)} \\
&\quad\cdot\int_{-\infty}^{\infty}\frac{dp}{-2\pi(ip+J_0^{\mathrm{tot}}+\frac{1}{2})}q^{\frac{1}{k}(p+\frac{ik}{2})^2-\frac{1}{4}-\frac{c_{\min}}{24}+L_0^{\mathrm{SL}(2,\mathbf{R})}+L_0^{N=2}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{\mathrm{U}(1)}+\frac{\kappa}{4}} \\
&\qquad\qquad\cdot\,\bar{q}^{\frac{1}{k}(p+\frac{ik}{2})^2-\frac{1}{4}-\frac{c_{\min}}{24}+\tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})}+\tilde{L}_0^{N=2}+\tilde{L}_0^{(\tilde{\nu})}+\tilde{L}_0^{(\tilde{\nu})}+\tilde{L}_0^{\mathrm{U}(1)}+\frac{\kappa}{4}} \\
&\quad-\mathrm{Tr}_{\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)}\otimes\mathcal{H}^{(\nu)}_{F_{l,2r}}\right)\otimes\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)}\otimes\mathcal{H}^{(\tilde{\nu})}_{F_{l,2r}}\right)} \\
&\quad\cdot\int_{-\infty}^{\infty}\frac{dp}{-2\pi(ip+J_0^{\mathrm{tot}}+\frac{1}{2})}q^{\frac{p^2}{k}-\frac{1}{4}-\frac{c_{\min}}{24}+L_0^{\mathrm{SL}(2,\mathbf{R})}+L_0^{N=2}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{\mathrm{U}(1)}} \\
&\qquad\qquad\cdot\,\bar{q}^{\frac{p^2}{k}-\frac{1}{4}-\frac{c_{\min}}{24}+\tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})}+\tilde{L}_0^{N=2}+\tilde{L}_0^{(\tilde{\nu})}+\tilde{L}_0^{(\tilde{\nu})}+\tilde{L}_0^{\mathrm{U}(1)}}\Bigg).
\end{aligned} \tag{5.55}$$

The first trace is simplified by replacing $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(-\frac{\kappa}{4},-\frac{\kappa}{2})}$'s with $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})}$'s, and at the same time removing $\frac{\kappa}{4}$ from the exponents of q and $\bar{q}$.

As was done in [13–15], we will now change the integration contour of the first trace from $p' \equiv p + \frac{ik}{2} \in \mathbf{R} + \frac{ik}{2}$ to $\mathbf{R}$. Then it picks up a residue of the pole at $p = i(J_0^{\mathrm{tot}} + \frac{1}{2})$ for the states satisfying $-\frac{k+1}{2} < J_0^{\mathrm{tot}} < -\frac{1}{2}$ (figure 4). We will show that in section 5.5 that these imaginary-momentum states reside below the lower bound of the continuous spectrum, and precisely on the boundary of the unitary region [44, 42]. That is, they are the discrete series representations.

5.3 The continuous spectrum

As in [13–15], we consider the continuous and discrete spectra separately.

The continuous spectrum arises from the first trace of (5.55) with the p-integration contour deformed, and the second trace for which we do not need any deformation. Since

$$\begin{aligned}
&q^{-\frac{1}{4}-\frac{c_{\min}}{24}+L_0^{\mathrm{SL}(2,\mathbf{R})}+L_0^{N=2}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{\mathrm{U}(1)}} \\
&\quad= q^{\left(L_0^{\mathrm{SL}(2,\mathbf{R})}-\frac{1}{8}\right)+\left(L_0^{N=2}-\frac{c_{\min}}{24}\right)+\left(L_0^{(\nu)}-\frac{1}{24}\right)+\left(L_0^{(\nu)}-\frac{1}{24}\right)+\left(L_0^{\mathrm{U}(1)}-\frac{1}{24}\right)},
\end{aligned} \tag{5.56}$$

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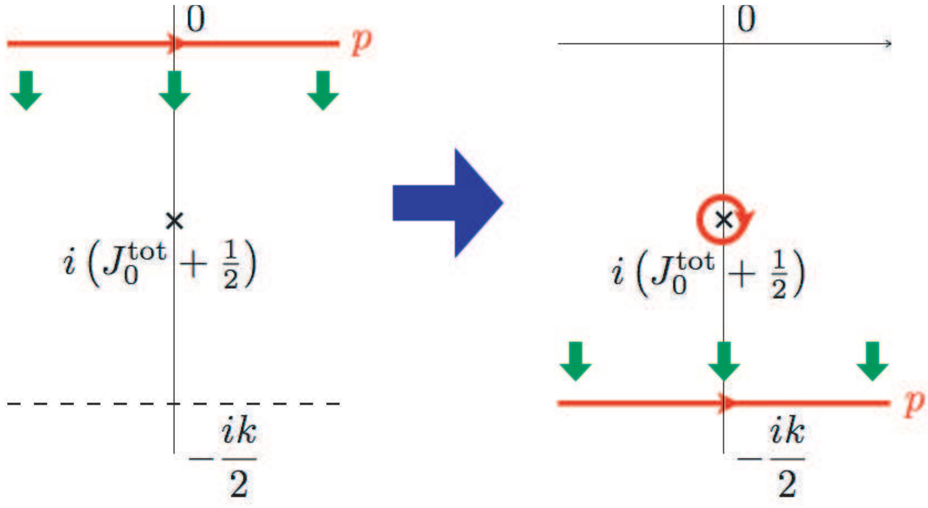

Figure 4: The contour deformation of the p integration.

if the denominator $ip + J_0^{\rm tot} + \frac{1}{2}$ were absent, we would formally obtain

$$\sum_\nu (-1)^\nu \mathrm{Tr}_{\mathcal{H}^{\rm SL(2,\mathbf{R})}_{\pm,(0,-\frac{\kappa}{2})}\otimes \mathcal{H}^{(\nu)}_{F_{l,2r}}} q^{-\frac{1}{4}-\frac{c_{\rm min}}{24}+L_0^{\rm SL(2,\mathbf{R})}+L_0^{N=2}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{\rm U(1)}} = \frac{\pm i}{\vartheta_1(\tau,0)}\cdot\frac{F_{l,2r}(\tau,0)}{\eta^3(\tau)}, \tag{5.57}$$

which contains a divergent factor $\frac{\pm i}{\vartheta_1(\tau,0)}$. This divergence comes from the zero mode contributions in the $\mathrm{SL}(2,\mathbf{R})$ module. In reality, the traces in the holomorphic and anti-holomorphic sectors are not independent but are constrained by the condition (5.52), but still the trace is divergent because, for a given pair of holomorphic and anti-holomorphic states with fixed values of L_0, $\tilde{L}_0$ and $J_0^{\rm tot}(=\tilde{J}_0^{\rm tot})$, there are infinitely many states having the same L_0, $\tilde{L}_0$ but different $J_0^{\rm tot}(=\tilde{J}_0^{\rm tot})$, and the sum of the form

$$-\sum_{n=0}^{\infty}\frac{1}{z-n} \tag{5.58}$$

does not converge. Following [14], we use the formula

$$-\sum_{n=0}^{\infty}\frac{e^{-n\epsilon}}{z-n} = -\log\epsilon + \frac{\partial}{\partial z}\log\Gamma(-z) - \mathcal{C} + O(\epsilon) + O(\epsilon\log\epsilon), \tag{5.59}$$

to regularize this divergence to obtain a finite answer. We give a proof for (5.59) in appendix D, thereby correcting (irrelevant) typos (the minus sign in front of log and Euler's constant) in [14]. Then the contribution to $-\frac{1}{2\pi(ip+J_0^{\rm tot}+\frac{1}{2})}$ from an arbitrary number of J_0^- multiplications in $\mathcal{H}^{\rm SL(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})}$ (in the first trace of (5.55)) is $-\frac{\log\epsilon}{2\pi}$ times

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}^{\rm SL(2,\mathbf{R})}_{-,(0,0)}/\{J_0^-\}} q^{L_0^{\rm SL(2,\mathbf{R})}} y^{J_0^3} &= \frac{-iq^{\frac{1}{8}}y^{+\frac{1}{2}}}{\vartheta_1(\tau,z)}\Big/\frac{1}{1-y^{-1}} \\ &\overset{z\to 0}{\longrightarrow} \frac{q^{\frac{1}{8}}}{\eta^3(\tau)}\end{aligned} \tag{5.60}$$

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to leading order. Here we denote by $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(h,j_0)}/\{J_0^-\}$ the coset of the module $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(h,l_0)}$ obtained by modding out the J_0^- multiplication. Similar equations hold for the second trace. Therefore, (5.55) becomes

$$
\begin{aligned}
(5.55) \;=\; \frac{-\log\epsilon}{2\pi}\cdot\frac{1}{4k}\int_{-\infty}^{\infty}dp\,(q\bar{q})^{\frac{p^2}{k}}\frac{1}{|\eta(\tau)|^2}\Bigg(&\frac{F_{l,2(r+1)}(\tau,0)(F_{\tilde{l},2(r+1)}(\tau,0))^*}{|\eta^3(\tau)|^2}\\
&+\frac{F_{l,2r}(\tau,0)(F_{\tilde{l},2r}(\tau,0))^*}{|\eta^3(\tau)|^2}\Bigg)+O(\epsilon^0).
\end{aligned}
\tag{5.61}
$$

$Z_{\mathcal{M}_4\times CY(X_n)}(\tau)$ (4.26) is obtained by summing (5.61) over $r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}$ and $l,\tilde{l}$ with a weight $N_{l,\tilde{l}}$, and hence

$$
\begin{aligned}
Z_{\mathcal{M}_4\times CY(X_n)}(\tau) &= C\frac{-\log\epsilon}{8\pi k}\sum_{l,\tilde{l}}N_{l,\tilde{l}}\sum_{r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}}\int_{-\infty}^{\infty}dp\,(q\bar{q})^{\frac{p^2}{k}}\frac{1}{|\eta(\tau)|^2} \qquad (5.62)\\
&\quad\cdot\left(\frac{F_{l,2(r+1)}(\tau,0)(F_{\tilde{l},2(r+1)}(\tau,0))^*}{|\eta^3(\tau)|^2}+\frac{F_{l,2r}(\tau,0)(F_{\tilde{l},2r}(\tau,0))^*}{|\eta^3(\tau)|^2}\right)+O(\epsilon^0)\\
&= C\frac{-\log\epsilon}{8\pi k}\sum_{l,\tilde{l}}N_{l,\tilde{l}}\sum_{r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}}\sqrt{\frac{k}{\tau_2}}\frac{1}{|\eta(\tau)|^2}\cdot\frac{F_{l,2r}(\tau,0)(F_{\tilde{l},2r}(\tau,0))^*}{|\eta^3(\tau)|^2}+O(\epsilon^0).
\end{aligned}
$$

This shows that the coefficient of the $\log\epsilon$ divergence of $Z_{\mathcal{M}_4\times CY(X_n)}(\tau)$ is precisely the integrand of the old partition function (3.26) (without the transverse boson factor) consisting of only the continuous series representations.

5.4 The discrete spectrum

Let us now consider the discrete spectrum, which is the main focus of this paper. In section 5.2, we have deformed the p-integration contour of the first trace in (5.55), the summation of which over l, $\tilde{l}$ (with a weight $N_{l,\tilde{l}}$) and r is equal to $Z_{\mathcal{M}_4\times CY(X_n)}(\tau)$. Then any state in $\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(-\frac{\kappa}{4},-\frac{\kappa}{2})}\otimes\mathcal{H}^{(\nu)}_{F_{l,2(r+1)}}\right)\otimes\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(-\frac{\kappa}{4},-\frac{\kappa}{2})}\otimes\mathcal{H}^{(\tilde{\nu})}_{F_{l,2(r+1)}}\right)$ such that the eigenvalue of J_0^{tot} is between $-\frac{k+1}{2}$ and $-\frac{1}{2}$ gives rise to a pole in the integrand. The resulting small contour around the pole is clock-wise, and the residue integral just cancels the $-2\pi i$ factor of the denominator. The residue contributions to $Z_{\mathcal{M}_4\times CY(X_n)}(\tau)$ are therefore

$$
\begin{aligned}
\text{Residues} = C\sum_{l,\tilde{l}}N_{l,\tilde{l}}&\sum_{r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}}\frac{|\eta^2(\tau)|^2}{4k} \qquad (5.63)\\
\cdot\sum_{\nu,\tilde{\nu}\in\mathbf{Z}_{4(k_{\min}+2)}}&(-1)^{\nu+\tilde{\nu}}\mathrm{Tr}_{\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})}\otimes\mathcal{H}^{(\nu)}_{F_{l,2r}}\right)\otimes\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})}\otimes\mathcal{H}^{(\tilde{\nu})}_{F_{l,2r}}\right)\Big|_{-\frac{k+1}{2}\le J_0^{\mathrm{tot}}\le\frac{1}{2},\; J_0^{\mathrm{tot}}=\tilde{J}_0^{\mathrm{tot}}}}\\
&\cdot(q\bar{q})^{\frac{1}{k}\left(i\left(J_0^{\mathrm{tot}}+\frac{1}{2}\right)+\frac{ik}{2}\right)^2}\\
&\cdot q^{\left(L_0^{\mathrm{SL}(2,\mathbf{R})}-\frac{1}{8}\right)+\left(L_0^{N=2}-\frac{c_{\min}}{24}\right)+\left(L_0^{(\nu)}-\frac{1}{24}\right)+\left(L_0^{(\nu)}-\frac{1}{24}\right)+\left(L_0^{\mathrm{U}(1)}-\frac{1}{24}\right)}\\
&\cdot\bar{q}^{\left(\tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})}-\frac{1}{8}\right)+\left(\tilde{L}_0^{N=2}-\frac{c_{\min}}{24}\right)+\left(\tilde{L}_0^{(\nu)}-\frac{1}{24}\right)+\left(\tilde{L}_0^{(\nu)}-\frac{1}{24}\right)+\left(\tilde{L}_0^{\mathrm{U}(1)}-\frac{1}{24}\right)},
\end{aligned}
$$

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where we have shifted $r+1$ to r in the r-summation, and also $L_0^{\mathrm{SL}(2,\mathbf{R})}$ (and $\tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})}$) by $\frac{\kappa}{4}$ as we mentioned below eq. (5.55).

As we noted at the beginning of section 4, (5.63) would become a polynomial with integer coefficients if we had started from the partition function (4.1) with $C = 4k$.

To obtain the discrete spectrum, we first relax the conditions for J_0^{tot} and $\tilde{J}_0^{\mathrm{tot}}$ and consider

$$\begin{aligned}
&\sum_{l,\tilde{l}} N_{l,\tilde{l}} \sum_{r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}} |\eta^2(\tau)|^2 \\
&\cdot \sum_{\nu,\tilde{\nu}\in\mathbf{Z}_{4(k_{\min}+2)}} (-1)^{\nu+\tilde{\nu}} \mathrm{Tr}_{\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})}\otimes\mathcal{H}^{(\nu)}_{F_{l,2r}}\right)\otimes\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})}\otimes\mathcal{H}^{(\tilde{\nu})}_{F_{\tilde{l},2r}}\right)} \\
&\qquad \cdot q^{\left(L_0^{\mathrm{SL}(2,\mathbf{R})}-\frac{1}{8}\right)+\left(L_0^{N=2}-\frac{c_{\min}}{24}\right)+\left(L_0^{(\nu)}-\frac{1}{24}\right)+\left(L_0^{(\nu)}-\frac{1}{24}\right)+\left(L_0^{\mathrm{U}(1)}-\frac{1}{24}\right)} y^{J_0^{\mathrm{tot}}} \\
&\qquad \cdot \bar{q}^{\left(\tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})}-\frac{1}{8}\right)+\left(\tilde{L}_0^{N=2}-\frac{c_{\min}}{24}\right)+\left(\tilde{L}_0^{(\nu)}-\frac{1}{24}\right)+\left(\tilde{L}_0^{(\nu)}-\frac{1}{24}\right)+\left(\tilde{L}_0^{\mathrm{U}(1)}-\frac{1}{24}\right)} \bar{y}^{\tilde{J}_0^{\mathrm{tot}}}
\end{aligned} \tag{5.64}$$

instead of (5.63). Next we find the states which satisfy the conditions $-\frac{k+1}{2} \leq J_0^{\mathrm{tot}} \leq -\frac{1}{2}$ and $J_0^{\mathrm{tot}} = \tilde{J}_0^{\mathrm{tot}}$, and then we take into account the "drop" of L_0 due to the imaginary momentum factor

$$(q\bar{q})^{\frac{1}{k}\left(i\left(J_0^{\mathrm{tot}}+\frac{1}{2}\right)+\frac{ik}{2}\right)^2} \tag{5.65}$$

in (5.63). Without the factor (5.65), we can easily evaluate (5.64):[9,10]

$$(5.64) = \sum_{l,\tilde{l}} N_{l,\tilde{l}} \sum_{r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}} \frac{2\hat{F}_{l,2r}(\tau,z)(2\hat{F}_{\tilde{l},2r}(\tau,z))^*}{\left|y^{\frac{\kappa-1}{2}}\tilde{\vartheta}_1(\tau,z)\eta(\tau)\right|^2}. \tag{5.66}$$

5.5 Massless spectra for odd $k_{\min}$

We consider the cases $k_{\min}$ odd and $k_{\min}$ even separately. We first assume that $k_{\min}$ is odd. Massless states in type II string theories come from those with the total conformal weight $\frac{1}{2}$. Therefore, they must lie at the lowest $L_0^{\mathrm{SL}(2,\mathbf{R})}$ level. Since J_0^3 takes values

$$J_0^3 = -\frac{\kappa}{2},\ -\frac{\kappa}{2}-1,\ -\frac{\kappa}{2}-2,\ldots \quad (\kappa = k+2) \tag{5.67}$$

at the lowest $L_0^{\mathrm{SL}(2,\mathbf{R})}$ level in $\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})}$, the condition

$$-\frac{k+1}{2} < J_0^{\mathrm{tot}} < -\frac{1}{2} \tag{5.68}$$

for the existence of a pole implies that a noncompact $N=2$ representation can contribute to the discrete series spectrum only if it carries a $J_0^{\mathrm{U}(1)}$ charge in the ranges

$$\frac{k_{\min}+4}{2} + (k_{\min}+4)n_{\mathrm{cluster}} < J_0^{\mathrm{U}(1)} < \left(\frac{k_{\min}+4}{2} + (k_{\min}+4)n_{\mathrm{cluster}}\right) + k_{\min}+2,$$

---

[9] $\hat{F}_{l,2r}(\tau,z)$ is so defined that $\hat{F}_{l,2r}(\tau,z)$ coincides with $F_{l,2r}(\tau,0)$, where the latter was defined in [13]. It is more natural to consider $2\hat{F}_{l,2r}$ here because it is a polynomial of q with integer coefficients.

[10] We should mention that in [68] a question has been raised as to whether the split of the partition function in this way is consistent with the degeneracy of descendent states of the $N=2$ superconformal algebra module. We leave this question open.

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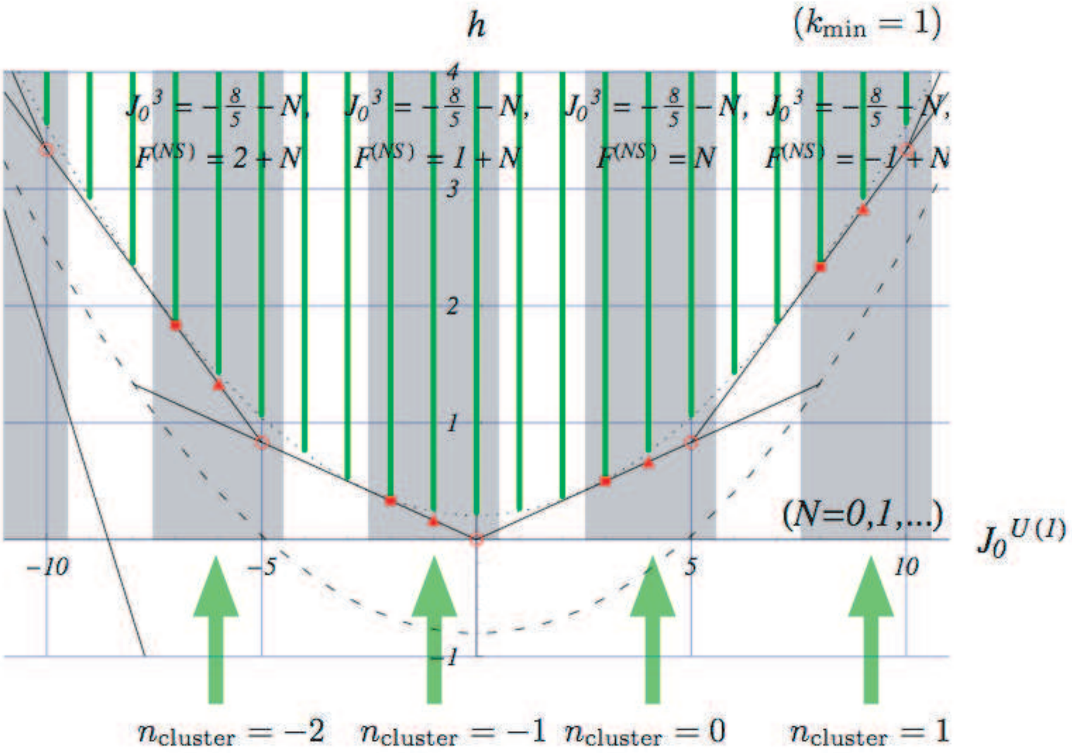

Figure 5: The clusters and the discrete states ($k_{\min} = 1$, NS sector). The circle at $(J_0^{\mathrm{U}(1)}, h) = (0, 0)$ in the $n_{\text{cluster}} = -1$ cluster and the square at $(3, \frac{1}{2})$ in the $n_{\text{cluster}} = 0$ cluster correspond to two massless supermultiplets for type II compactifications.

$$n_{\text{cluster}} \equiv -\frac{\kappa}{2} - J_0^3 - F^{(\nu)} \in \begin{cases} \mathbf{Z} & \text{(NS sector)}, \\ \mathbf{Z} + \frac{1}{2} & \text{(R sector)}, \end{cases} \tag{5.69}$$

where we have introduced a label n_{cluster} to distinguish different "clusters" of relevant noncompact $N = 2$ representations (figure 5).

Let us consider a continuous family of noncompact $N = 2$ representations with a definite $J_0^{\mathrm{U}(1)}$ charge in the range (5.69), which is drawn as a semi-infinite line in figure 5. As we discussed, such a family in the partition function is accompanied by a residue contribution, which has a conformal weight yet lower than the lower bound of the continuous spectrum by an amount equal to the exponent of (5.65). For the holomorphic part, it is found to be

$$-\frac{\left(J_0^{\mathrm{U}(1)} - (k_{\min} + 4)(n_{\text{cluster}} + \frac{1}{2})\right)^2}{2(k_{\min} + 2)(k_{\min} + 4)}. \tag{5.70}$$

The lower bound of the continuous spectrum (which can be read off from the level-$2(k_{\min} + 2)(k_{\min} + 4)$ theta function) is

$$\frac{1}{4} - \frac{c_{\min}}{24} + 2(k_{\min} + 2)(k_{\min} + 4)\left(\frac{J_0^{\mathrm{U}(1)}}{2(k_{\min} + 2)(k_{\min} + 4)}\right)^2. \tag{5.71}$$

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Adding (5.70) to (5.71), we find the conformal weight of the residue contribution

$$(5.71) + (5.70) = \frac{\left(n_{\text{cluster}} + \frac{1}{2}\right) J_0^{\text{U}(1)}}{k_{\min} + 2} - \frac{\hat{c}_{\text{KS}} - 1}{2} \left(\left(n_{\text{cluster}} + \frac{1}{2} \right)^2 - \frac{1}{4} \right), \tag{5.72}$$

where

$$\hat{c}_{\text{KS}} = \frac{\kappa}{\kappa - 2} \tag{5.73}$$

is $\frac{1}{3}$ of the central charge of the noncompact $N = 2$ CFT. (5.72) is precisely the series of equations of the boundary lines surrounding the polygonal region of the $N = 2$ unitary representations [44] (figure 5). Thus we have shown that they are indeed the $N = 2$ representations coming from the discrete series of $\text{SL}(2, \mathbf{R})$.

It turns out that the states with the total conformal weight $\frac{1}{2}$ exist only in the $n_{\text{cluster}} = 0$ and $n_{\text{cluster}} = -1$ clusters. If $n_{\text{cluster}} = 0$, (5.67) implies that the "Liouville fermion number" (that is, the number of the fermion oscillators of the noncompact $N = 2$ CFT) in the NS sector $F^{(\nu)}$ $(= F^{(\text{NS})})$ takes values $0, 1, \ldots$. (The R sector can be analyzed in the same way as done below; anyway the supersymmetry ensures that the massless spectra must be identical.) For massless states $F^{(\text{NS})}$ must be 0 or 1 because otherwise the conformal weight h exceeds $\frac{1}{2}$. The noncompact $N = 2$ CFT representations in the $n_{\text{cluster}} = 0$ cluster have $J_0^{\text{U}(1)}$ charges in the range

$$\frac{k_{\min}}{2} + 2 < \ J_0^{\text{U}(1)} \ < \frac{3k_{\min}}{2} + 4. \tag{5.74}$$

The upper limit for massless states is much stronger than this:

$$\frac{k_{\min}}{2} + 2 < \ J_0^{\text{U}(1)} \ \leq k_{\min} + 2 \tag{5.75}$$

because otherwise the straight-line boundary, on which the discrete series resides, already goes above $h = \frac{1}{2}$. Therefore, there are $\frac{k_{\min}+1}{2}$ different possible $J_0^{\text{U}(1)}$ charges

$$J_0^{\text{U}(1)} = k_{\min} + 2 - j \quad \left(j = 0, \ldots, \frac{k_{\min} - 1}{2} \right). \tag{5.76}$$

Since it does not contain $J_0^{\text{U}(1)} = 0$, $F^{(\text{NS})}$ needs to be 0 and we look for $h = \frac{1}{2}$ combinations of states of the noncompact coset and $N = 2$ minimal CFT sectors. It turns out that for every j above, there exists precisely one $\hat{F}_{j,2r}$ which contains an $h = \frac{1}{2}$ combination; this is $\hat{F}_{j,j+2}$. Indeed, it contains NS-sector terms such as (See appendix.)

$$\begin{aligned} \hat{F}^{(+)\text{NS}}_{j,j+2}(\tau, z) \cdot q^{\frac{1}{4}} &= q^{\frac{1}{4}} \chi_j^{j,0}(\tau, 0) (\Theta_{0,2}(\tau, 0) \Theta_{0,2}(\tau, z) + \Theta_{2,2}(\tau, 0) \Theta_{2,2}(\tau, z)) \\ &\qquad \cdot \Theta_{-2j+2(k_{\min}+2), 2(k_{\min}+2)(k_{\min}+4)} \left(\tau, \frac{z}{k_{\min} + 4} \right) + \cdots \\ &= q^{\frac{1}{4} - \frac{c_{\min}}{24} + \frac{j}{2(k_{\min}+2)} + \frac{(-j+k_{\min}+2)^2}{2(k_{\min}+2)(k_{\min}+4)}} y^{\frac{-j+k_{\min}+2}{k_{\min}+4}} + \cdots, \end{aligned} \tag{5.77}$$

where we have taken into account the extra factor of $q^{\frac{1}{4}}$ because of the shortage of the eta (or theta) functions[11] in the denominator of the partition function. If we include the imaginary-momentum contribution (5.70) with $n_{\rm cluster}=0$, we have the conformal weight

$$\frac{1}{4}-\frac{c_{\rm min}}{24}+\frac{j}{2(k_{\rm min}+2)}+\frac{(-j+k_{\rm min}+2)^2}{2(k_{\rm min}+2)(k_{\rm min}+4)}-\frac{\left(-j+k_{\rm min}+2-\frac{k_{\rm min}+4}{2}\right)^2}{2(k_{\rm min}+2)(k_{\rm min}+4)}=\frac{1}{2}. \tag{5.78}$$

Since $\hat{F}_{j,j+2}=\hat{F}_{k_{\rm min}-j,k_{\rm min}-j+2}$ (B.42), $\hat{F}^{(+){\rm NS}}_{l,l+2}$ give rise to an $h=\frac{1}{2}$ state for every $l=0,1,\ldots,k_{\rm min}$. The character of the $N=2$ minimal model is $\chi^{j,0}_j$. The anti-holomorphic sector is similar. So for the $A_{k_{\rm min}+1}$ modular invariant there are $\frac{k_{\rm min}+1}{2}$ complex scalars in the NS-NS sector.

Taking also the Ramond sector into account, each NS-NS complex scalar becomes a part of a single hyper-multiplet for type IIA (since the two massless Ramond-Ramond states are spacetime scalars), and a single vector multiplet for typeIIB strings (since the Ramond-Ramond states become the two helicity states of a massless vector).

Next we turn to the $n_{\rm cluster}=-1$ cluster. In this case (5.67) and (5.69) require that the NS-sector Liouville fermion number $F^{\rm (NS)}$ takes values ≥ 1. This means that the Liouville fermion already “spends” the maximal conformal weight for massless states (that is, $h=\frac{1}{2}$) so that the remaining CFT sectors can only associate an $h=0$ state with it. In this $n_{\rm cluster}=-1$ cluster, $J_0^{\rm U(1)}$ takes integral values in the range

$$-\frac{k_{\rm min}}{2}-2< \ J_0^{\rm U(1)} \ <\frac{k_{\rm min}}{2}, \tag{5.79}$$

and indeed contains $J_0^{\rm U(1)}=0$. The family of noncompact $N=2$ representations with $J_0^{\rm U(1)}=0$ are contained in several $\hat{F}_{l,2r}$'s, among which

$$\begin{aligned}\hat{F}^{(-){\rm NS}}_{0,0}(\tau,z)\cdot q^{\frac{1}{4}} \ &= q^{\frac{1}{4}}\chi^{0,0}_0(\tau,0)(\Theta_{0,2}(\tau,0)\Theta_{2,2}(\tau,z)+\Theta_{2,2}(\tau,0)\Theta_{0,2}(\tau,z))\\ &\qquad\cdot\Theta_{0,2(k_{\rm min}+2)(k_{\rm min}+4)}\left(\tau,\frac{z}{k_{\rm min}+4}\right)+\cdots\end{aligned} \tag{5.80}$$

only gives rise to an $h=0$ combination of states of the noncompact and minimal $N=2$ CFTs, if the effect (5.70) of the imaginary-momentum factor is taken into account. Since $J_0^{\rm U(1)}=0$ for the $n_{\rm cluster}=-1$ cluster of $\hat{F}^{\rm NS}_{0,0}(\tau,z)$, (5.70) becomes

$$(5.70)=-\frac{k_{\rm min}+4}{8(k_{\rm min}+2)}, \tag{5.81}$$

[11] To read off the conformal weights of the internal CFT representations from (5.66), we write the denominator as an integer power series of q with an overall ghost ground state factor (in the spherical worldsheet coordinates) of $q^{\frac{1}{2}}$. In the usual critical strings, this factor may be thought of as provided by the 12 eta functions coming from the 8 transverse bosons and the 4 complex fermions, that is, the normal ordering constant factor in the cylindrical worldsheet coordinates. In the present case, we have only one ϑ_1 and three η's (one from (5.66) and two from the transverse bosons (4.9)), so we need to multiply both the denominator and the numerator by $q^{\frac{1}{4}}$. This is the Liouville energy, which makes the tachyon be massless in two-dimensional string theory.

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which cancels the extra conformal weight of the continuous series $\frac{1}{4}-\frac{c_{\min}}{24}$. The first level-2 theta function with $z=0$ comes from the complex fermion of the transverse spacetime dimensions, while the second level-2 theta function is the one from the Liouville fermion. $\Theta_{2,2}(\tau,0)\Theta_{0,2}(\tau,z)$ contains a term $q^{\frac{1}{2}}(y^{+1}+y^{-1})$, of which $q^{\frac{1}{2}}y^{+1}$ has $F^{(\mathrm{NS})}=1$ and corresponds to a discrete state. On the other hand, although $\Theta_{2,2}(\tau,0)\Theta_{0,2}(\tau,z)$ has $2q^{\frac{1}{2}}$ in the expansion, they have $F^{(\mathrm{NS})}=0$ and hence do not correspond to discrete states. Combined with a similar state in the anti-holomorphic sector, this $h=\frac{1}{2}$ state becomes a real scalar in the four-dimensional spacetime. Since $\hat{F}_{0,0}=\hat{F}_{k_{\min},2(k_{\min}+4)}$, there is another real scalar.

Taking account of the Ramond sector again, they constitute a single hyper/vector multiplet for typeIIA/IIB strings. In all, for the $A_{k_{\min}+1}$ modular invariant with odd $k_{\min}$, there are $\frac{k_{\min}+3}{2}$ massless hyper/vector multiplets for typeIIA/IIB strings. $\frac{k_{\min}+1}{2}$ of them have NS-sector discrete states in the $n_{\mathrm{cluster}}=0$ cluster, whereas one has those in the $n_{\mathrm{cluster}}=-1$ cluster. R-sector discrete states are all in the $n_{\mathrm{cluster}}=-\frac{1}{2}$ cluster.

In usual "compact" Gepner models, where the internal $N=2$ CFT consists of only the $N=2$ minimal models, the internal $h=0$ state is always accompanied by a graviton, an anti-symmetric tensor and a dilaton, with their superpartners. In contrast, we have only a massless scalar in the spectrum and there is no localized massless graviton due to the constraint $F^{(\mathrm{NS})}\geq 1$ for $n_{\mathrm{cluster}}=-1$.[12]

We should also note that, although the massless state in the $n_{\mathrm{cluster}}=-1$ cluster comes from the free boson module $\mathcal{H}_{0,K}$, it does not contain the identity representation module of the noncompact $N=2$ CFT because that massless state is made of a combination of $|0\rangle$ in the free boson module *and* an $F^{(\mathrm{NS})}=1$ state in the free fermion module. This combination of states has $h=\frac{1}{2}$ and hence is not contained in the $N=2$ identity representation module. This can be seen by the fact that the generic (reducible) $N=2$ character with $h=0$, $Q=0$ is decomposed into irreducible characters of the identity representation and two discrete series representations with $h=\frac{1}{2}$, $Q=\pm 1$. This is consistent with the fact that the identity representation of $\mathrm{SL}(2,\mathbf{R})$ does not correspond to a normalizable mode.

5.6 Massless spectra for even $k_{\min}$: a gapless[13] spectrum

Massless spectra for even $k_{\min}$ are similar to those for the odd case, but there is a crucial difference. After the contour deformation discussed in section 5.4, some families of continuous series "leave behind" discrete series as pole contributions if

$$\frac{k_{\min}+4}{2}+(k_{\min}+4)n_{\mathrm{cluster}}\leq J_0^{\mathrm{U}(1)}\leq\left(\frac{k_{\min}+4}{2}+(k_{\min}+4)n_{\mathrm{cluster}}\right)+k_{\min}+2, \quad (5.82)$$

where n_{cluster} is given by (5.69). Like in the case of odd $k_{\min}$, only the $n_{\mathrm{cluster}}=-1$ and 0 clusters are relevant for the massless spectrum. In the $n_{\mathrm{cluster}}=0$ cluster in the NS sector, massless states only come from noncompact $N=2$ representations with $h\leq\frac{1}{2}$, so the

[12]The author thanks T.Eguchi and Y.Sugawara for discussions on this point.

[13]Here by "gap" we mean (1) literally an opening between the end of a continuous spectrum and a discrete state lying on a segment, and (2) a mass of a state. In both senses there is no gap for the states indicated by arrows in figure 6.

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upper limit of (5.69) is lowered to

$$\frac{k_{\min}+4}{2} \leq J_0^{\mathrm{U}(1)} \leq k_{\min}+2. \tag{5.83}$$

Since $k_{\min}$ is even, the contours before and after the deformation can "hit" the pole if $J_0^{\mathrm{U}(1)}$ is at either of the two ends of the domain (5.82). For the restricted range (5.83), the pole can still be located *on* the contour *after* the deformation if $J_0^{\mathrm{U}(1)}$ is at the lower limit ($= \frac{k_{\min}+4}{2}$). This means that the continuous spectrum already reaches the boundary of the unitary region (figure 6). Depending on how the contour is deformed to circumvent this pole, the residue may or may not contribute to the partition function, and irrespective of how it is deformed, there exist a continuous spectrum of modes arbitrarily close to the discrete mode.

As we discussed in [8], a generic $N = 2$ representation becomes reducible at the boundary of the unitary region, where the generic character is decomposed into a sum of characters of discrete (including the identity) representations. Also, in that paper we interpreted this massless state as the geometric modulus of the conifold. In this paper, we regard the geometric moduli of a singular Calabi-Yau not as a part of the continuous spectra, but as pole contributions to the partition function. In the conifold ($k_{\min} = 0$) case, there is another massless multiplet from the $n_{\text{cluster}} = -1$ sector, which has a nonzero mass gap below the lowest end of the continuum and hence may be identified as the geometric modulus. Since the (deformed) conifold has only one modulus (the size of the S^3), this would imply that the gapless state does not correspond to any topological cycle for general even-$k_{\min}$ models.

Besides the gapless spectrum at $J_0^{\mathrm{U}(1)} = \frac{k_{\min}+4}{2}$, there are $\frac{k_{\min}}{2}$ possible $J_0^{\mathrm{U}(1)}$ values

$$J_0^{\mathrm{U}(1)} = k_{\min}+2-j \quad (j = 0,\ldots,\frac{k_{\min}}{2}-1). \tag{5.84}$$

which give rise to massless states, similarly to the $k_{\min}$ odd case. Again $\hat{F}_{j,j+2}$ and $\hat{F}_{k_{\min}-j,k_{\min}-j+2}$ ($j = 0,\ldots,\frac{k_{\min}}{2}-1$) correspond to such states. The all reside below the lowest limit of the continuous spectrum with a finite mass gap. The $n_{\text{cluster}} = -1$ sector is also similar to that for the $k_{\min}$ odd case.

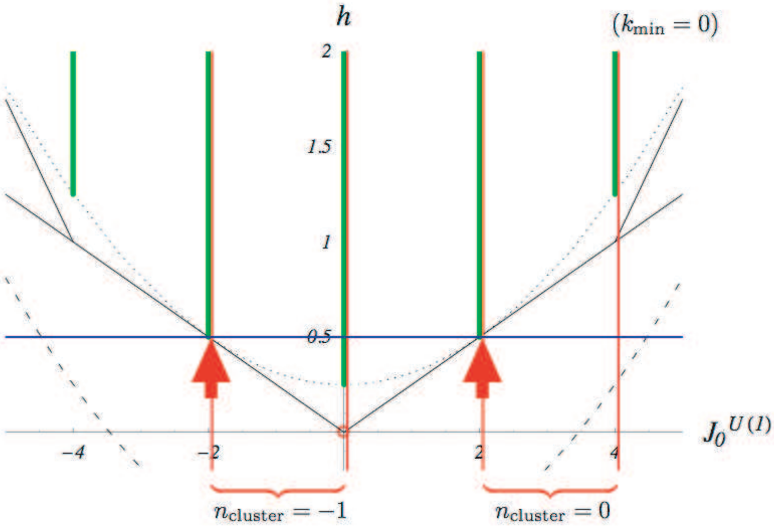
To summarize, for the $A_{k_{\min}+1}$ modular invariant with even $k_{\min}$, there are (excluding the gapless one) $\frac{k_{\min}}{2}+1$ massless hyper/vector multiplets for typeIIA/IIB strings. Similarly to the $k_{\min}$ odd case, $\frac{k_{\min}}{2}$ of them has NS-sector discrete states in the $n_{\text{cluster}} = 0$ cluster, and one has those in the $n_{\text{cluster}} = -1$.

We should note that the pattern of the chiral ring structure has already been recognized in [13]. The recognition of the gapless spectrum for the even $k_{\min}$ case is new, however.

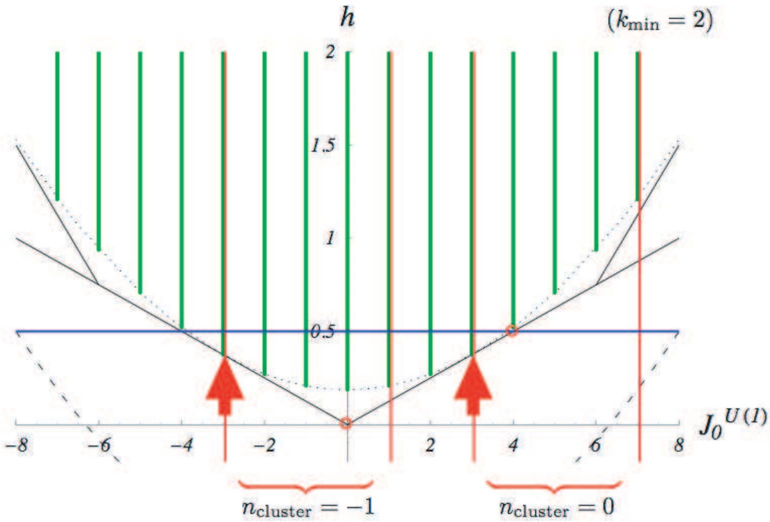
5.7 Separation of the discrete series for heterotic strings

Massless discrete spectra for heterotic strings can be similarly obtained from the heterotic conversion of (5.66):

$$\longrightarrow \sum_{l,\tilde{l}} N_{l,\tilde{l}} \sum_{r\in\mathbf{Z}_{k_{\min}+4}+\frac{l}{2}} \frac{2\hat{F}^{\mathrm{het}}_{l,2r}(\tau,z)(2\hat{F}_{\tilde{l},2r}(\tau,z))^*}{\left|y^{\frac{\kappa-1}{2}}\tilde{\vartheta}_1(\tau,z)\right|^2 \eta^{13}(\tau)(\eta(\tau))^*}. \tag{5.85}$$



(a) The $k_{\min} = 0$ case.



(b) The $k_{\min} = 2$ case.

Figure 6: Spectra for even $k_{\min}$. The small circles show the locations of the massless discrete states. Some continuous spectra reach the boundary of the unitary region (the arrows). The $N = 2$ U(1) charge Q is $= \frac{J_0^{\mathrm{U}(1)}}{k_{\min}+2}$.

where $\hat{F}^{\text{het}}_{l,2r}$ is $\hat{F}^{E_8\times E_8}_{l,2r}$ or $\hat{F}^{\mathrm{SO}(32)}_{l,2r}$ given in appendix. In this case we search for $h = 1$ states for the left (holomorphic) sector.

$h = 1$ states in $\hat{F}^{\text{het}}_{j,j+2}$ with odd $k_{\min}$

As we did in the type II case, we first assume that $k_{\rm min}$ is odd. We have seen in the previous sections that $\hat{F}^{(+){\rm NS}}_{j,j+2}(\tau,z)$ $(j=0,\ldots,\frac{k_{\rm min}}{2}-1)$ has $h=\frac{1}{2}$ discrete states:

$$\begin{aligned}
&\hat{F}^{(+){\rm NS}}_{j,j+2}(\tau,z)\cdot q^{\frac{1}{4}-\frac{\left(-j+k_{\rm min}+2-\frac{k_{\rm min}+4}{2}\right)^2}{2(k_{\rm min}+2)(k_{\rm min}+4)}}\\
&=q^{\frac{1}{4}-\frac{\left(-j+k_{\rm min}+2-\frac{k_{\rm min}+4}{2}\right)^2}{2(k_{\rm min}+2)(k_{\rm min}+4)}}\left(\chi^{j,0}_j(\tau,0)\Theta_{-2j+2(k_{\rm min}+2),2(k_{\rm min}+2)(k_{\rm min}+4)}\left(\tau,\frac{z}{k_{\rm min}+4}\right)\right.\\
&\qquad\left.+\chi^{k_{\rm min}-j,0}_{-(k_{\rm min}-j)}(\tau,0)\Theta_{-2j-4,2(k_{\rm min}+2)(k_{\rm min}+4)}\left(\tau,\frac{z}{k_{\rm min}+4}\right)+\cdots\right)\\
&\qquad\cdot\left(\Theta_{0,2}(\tau,0)\Theta_{0,2}(\tau,z)+\Theta_{2,2}(\tau,0)\Theta_{2,2}(\tau,z)\right), \qquad (5.86)
\end{aligned}$$

where we have written out the $m=j+2$ (as well as the $m=j$) term in (B.34) because it becomes relevant for the heterotic massless spectrum. Also, we have already included the factor from the imaginary momentum. If it is converted to the heterotic versions, the level-2 theta functions with argument $(\tau,0)$ change as

$$\Theta_{0,2}=\frac{\vartheta_3+\vartheta_4}{2}\to\frac{(\vartheta_3)^5-(\vartheta_4)^5}{2\eta^4}B^{(E_8)}, \qquad (5.87)$$

$$\Theta_{2,2}=\frac{\vartheta_3-\vartheta_4}{2}\to\frac{(\vartheta_3)^5+(\vartheta_4)^5}{2\eta^4}B^{(E_8)}, \qquad (5.88)$$

and also for the SO(32) theory as

$$\Theta_{0,2}=\frac{\vartheta_3+\vartheta_4}{2}\to\frac{(\vartheta_3)^{13}-(\vartheta_4)^{13}}{2\eta^{12}}, \qquad (5.89)$$

$$\Theta_{2,2}=\frac{\vartheta_3-\vartheta_4}{2}\to\frac{(\vartheta_3)^{13}+(\vartheta_4)^{13}}{2\eta^{12}}. \qquad (5.90)$$

The $h=\frac{1}{2}$ states come from the lowest term of

$$\frac{\vartheta_3+\vartheta_4}{2}(\tau,0)\frac{\vartheta_3+\vartheta_4}{2}(\tau,z)+\frac{\vartheta_3-\vartheta_4}{2}(\tau,0)\frac{\vartheta_3-\vartheta_4}{2}(\tau,z)=1+\cdots, \qquad (5.91)$$

which is converted to (besides the eta functions)

$$\frac{\vartheta_3^5-\vartheta_4^5}{2}(\tau,0)\frac{\vartheta_3+\vartheta_4}{2}(\tau,z)+\frac{\vartheta_3^5+\vartheta_4^5}{2}(\tau,0)\frac{\vartheta_3-\vartheta_4}{2}(\tau,z)=10q^{\frac{1}{2}}+q^{\frac{1}{2}}(y+y^{-1})+\cdots \qquad (5.92)$$

for the $E_8\times E_8$ case ($B^{(E_8)}=1+\cdots$), and

$$\frac{\vartheta_3^{13}-\vartheta_4^{13}}{2}(\tau,0)\frac{\vartheta_3+\vartheta_4}{2}(\tau,z)+\frac{\vartheta_3^{13}+\vartheta_4^{13}}{2}(\tau,0)\frac{\vartheta_3-\vartheta_4}{2}(\tau,z)=26q^{\frac{1}{2}}+q^{\frac{1}{2}}(y+y^{-1})+\cdots \qquad (5.93)$$

for the SO(32) case.

As we have shown in (5.78), the first term on the right hand side of (5.86) starts from $q^{\frac{1}{2}}y^{\frac{-j+k_{\rm min}+2}{k_{\rm min}+4}}$. This is in the $n_{\rm cluster}=0$ cluster. Either a transverse or a Liouville fermion may be excited. Therefore, (5.92) shows that there are $(10+1)$ $h=1$ states in the $E_8\times E_8$ theory; the latter singlet comes from $q^{\frac{1}{2}}y$ which has $F^{(\mathrm{NS})}=+1$, while $q^{\frac{1}{2}}y^{-1}$ does not corresponds to a discrete state because $F^{(\mathrm{NS})}=-1$ is not allowed in the $n_{\text{cluster}}=0$ cluster. Similarly, we can see from (5.93) that there are $(26+1)$ $h=1$ states in the SO(32) theory.

On the other hand, the second term of (5.86) has also an expansion $q^{\frac{1}{2}}y^{\frac{-j-2}{k_{\min}+4}}+\cdots$. This is in the $n_{\text{cluster}}=-1$ cluster, and therefore it did not produce any massless states in typeII theories. However, with a Liouville fermion excitation, it is allowed in heterotic theories. This gives another $h=1$ state for both the $E_8\times E_8$ and SO(32) theories.

Next consider the R-sector terms of $\hat{F}^{(+)}_{j,j+2}(\tau,z)$:

$$\begin{aligned}
&\hat{F}^{(+)\mathrm{R}}_{j,j+2}(\tau,z)\cdot q^{\frac{1}{4}-\frac{\left(-j+k_{\min}+2-\frac{k_{\min}+4}{2}\right)^2}{2(k_{\min}+2)(k_{\min}+4)}}\\
&=q^{\frac{1}{4}-\frac{\left(-j+k_{\min}+2-\frac{k_{\min}+4}{2}\right)^2}{2(k_{\min}+2)(k_{\min}+4)}}\left(\chi^{j,1}_{j+1}(\tau,0)\Theta_{-2j+k_{\min},2(k_{\min}+2)(k_{\min}+4)}\left(\tau,\frac{z}{k_{\min}+4}\right)+\cdots\right)\\
&\qquad\cdot\left(\Theta_{1,2}(\tau,0)\Theta_{1,2}(\tau,z)+\Theta_{-1,2}(\tau,0)\Theta_{-1,2}(\tau,z)\right)\\
&=\left(q^{\frac{1}{4}}y^{\frac{-j+\frac{k_{\min}}{2}}{k_{\min}+4}}+\cdots\right)\left(\Theta_{1,2}(\tau,0)\Theta_{1,2}(\tau,z)+\Theta_{-1,2}(\tau,0)\Theta_{-1,2}(\tau,z)\right).
\end{aligned}\tag{5.94}$$

This is in the $n_{\text{cluster}}=-\frac{1}{2}$ cluster, and so the $y^{+\frac{1}{2}}$ terms survive. Through the heterotic conversion,

$$\begin{aligned}
\frac{\vartheta_2+\tilde{\vartheta}_1}{2}(\tau,0)\frac{\vartheta_2+\tilde{\vartheta}_1}{2}(\tau,z)+\frac{\vartheta_2-\tilde{\vartheta}_1}{2}(\tau,0)\frac{\vartheta_2-\tilde{\vartheta}_1}{2}(\tau,z)&=q^{\frac{1}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots\\
(=\ \Theta_{1,2}(\tau,0)\Theta_{1,2}(\tau,z)+\Theta_{-1,2}(\tau,0)\Theta_{-1,2}(\tau,z))&
\end{aligned}\tag{5.95}$$

is replaced with

$$\frac{\vartheta_2^5+\tilde{\vartheta}_1^5}{2}(\tau,0)\frac{\vartheta_2+\tilde{\vartheta}_1}{2}(\tau,z)+\frac{\vartheta_2^5-\tilde{\vartheta}_1^5}{2}(\tau,0)\frac{\vartheta_2-\tilde{\vartheta}_1}{2}(\tau,z)=16q^{\frac{3}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots\tag{5.96}$$

in the $E_8\times E_8$ case, or

$$\frac{\vartheta_2^{13}+\tilde{\vartheta}_1^{13}}{2}(\tau,0)\frac{\vartheta_2+\tilde{\vartheta}_1}{2}(\tau,z)+\frac{\vartheta_2^{13}-\tilde{\vartheta}_1^{13}}{2}(\tau,0)\frac{\vartheta_2-\tilde{\vartheta}_1}{2}(\tau,z)=2^{12}q^{\frac{7}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots\tag{5.97}$$

in the SO(32) case. (5.96) shows that there are sixteen $h=1$ states in the Ramond sector of the $E_8\times E_8$ theory, while (5.97) implies no $h=1$ states in the Ramond sector of the SO(32) theory.

Summarizing the $h=1$ states in $\hat{F}^{\text{het}}_{j,j+2}$ $(j=0,\ldots,\frac{k_{\min}}{2}-1)$, $\hat{F}^{E_8\times E_8}_{j,j+2}$ has

$$\mathbf{10}\oplus\mathbf{1}\oplus\mathbf{1}\ \text{(NS sector)},\quad \mathbf{16}\ \text{(R sector)}$$

of SO(10), while $\hat{F}^{\mathrm{SO}(32)}_{j,j+2}$ has

$$\mathbf{26}\oplus\mathbf{1}\oplus\mathbf{1}\ \text{(NS sector)},\quad \text{no states (R sector)}$$

of SO(26). Taking into account the right moving part and also the symmetry $\hat{F}^{\text{het}}_{j,j+2}=\hat{F}^{\text{het}}_{k_{\min}-j,k_{\min}-j+2}$, they become $D=4$, $N=1$ chiral supermultiplets.

$h = 1$ *states in* $\hat{F}^{\rm het}_{0,0}$ *with odd* $k_{\rm min}$

Just like the type II case, there are also $h = 1$ states that contribute to $\hat{F}^{\rm het}_{0,0}$. Before the conversion, the NS-sector terms are

$$\begin{aligned}
&\hat{F}^{(-){\rm NS}}_{0,0}(\tau,z)\cdot q^{\frac{1}{4}-\frac{k_{\rm min}+4}{8(k_{\rm min}+2)}}\\
&= q^{\frac{1}{4}-\frac{k_{\rm min}+4}{8(k_{\rm min}+2)}}\left(\chi^{0,0}_0(\tau,0)\Theta_{0,2(k_{\rm min}+2)(k_{\rm min}+4)}\left(\tau,\frac{z}{k_{\rm min}+4}\right)\right.\\
&\qquad\left.+\chi^{0,-2}_{-2}(\tau,0)\Theta_{+2(k_{\rm min}+4),2(k_{\rm min}+2)(k_{\rm min}+4)}\left(\tau,\frac{z}{k_{\rm min}+4}\right)+\cdots\right)\\
&\qquad\qquad\cdot\left(\Theta_{0,2}(\tau,0)\Theta_{2,2}(\tau,z)+\Theta_{2,2}(\tau,0)\Theta_{0,2}(\tau,z)\right).
\end{aligned} \tag{5.98}$$

$\hat{F}^{(+){\rm NS}}_{0,0}(\tau,z)$ gives rise to no $h = 1$ states and hence is not written here. We again included the Liouville energy and the imaginary momentum factor. As we saw in the type II analysis, this cancels the $q^{-\frac{c_{\rm min}}{24}}$ of the $N = 2$ characters, giving

$$= \left(q^0y^0+\cdots+q^1y^1+\cdots\right)\left(\frac{\vartheta_3+\vartheta_4}{2}(\tau,0)\frac{\vartheta_3-\vartheta_4}{2}(\tau,z)+\frac{\vartheta_3-\vartheta_4}{2}(\tau,0)\frac{\vartheta_3+\vartheta_4}{2}(\tau,z)\right). \tag{5.99}$$

In the $E_8\times E_8$ case, this is converted to

$$\begin{aligned}
&\to\left(q^0y^0+\cdots+q^1y^1+\cdots\right)\left(\frac{\vartheta_3^5-\vartheta_4^5}{2}(\tau,0)\frac{\vartheta_3-\vartheta_4}{2}(\tau,z)+\frac{\vartheta_3^5+\vartheta_4^5}{2}(\tau,0)\frac{\vartheta_3+\vartheta_4}{2}(\tau,z)\right).\\
&=\left(q^0y^0+\cdots+q^1y^1+\cdots\right)\left(10q^1(y^{+1}+y^{-1})+\cdots+1+\cdots\right).
\end{aligned} \tag{5.100}$$

The q^0y^0 term corresponds to a state in the $n_{\rm cluster} = -1$ cluster. Therefore, only terms with $F^{(NS)} \geq 1$ (that is, those containing y^{+1} as a factor in the second parenthesis) are relevant for the discrete spectrum, *as long as* $L_0^{{\rm SL}(2,\mathbf{R})} = 0$. This gives **10**. While states at $L_0^{{\rm SL}(2,\mathbf{R})} = 0$ in the module $\mathcal{H}^{{\rm SL}(2,\mathbf{R})}_{\pm,(0,-\frac{\kappa}{2})}$ have J_0^3 charges

$$J_0^3 = -\frac{\kappa}{2},\quad -\frac{\kappa}{2}-1,\quad -\frac{\kappa}{2}-2,\ldots, \tag{5.101}$$

those at $L_0^{{\rm SL}(2,\mathbf{R})} = 1$ have

$$J_0^3 = -\frac{\kappa}{2}+1,\quad -\frac{\kappa}{2},\quad -\frac{\kappa}{2}-1,\ldots. \tag{5.102}$$

Therefore, for states at $L_0^{{\rm SL}(2,\mathbf{R})} = 1$, the condition $F^{(NS)} \geq -n_{\rm cluster}$ is relaxed to $F^{(NS)} \geq -n_{\rm cluster} - 1 = 0$. In this case, q^0y^0 can also be paired with "1", with a total conformal weight $h = 1$ due to $L_0^{{\rm SL}(2,\mathbf{R})} = 1$. This is a singlet.

The q^1y^1 term is in the $n_{\rm cluster} = 0$ cluster. This can be paired with "1" and gives rise to another singlet of SO(10).

In the SO(32) case, (5.99) becomes

$$\to\left(q^0y^0+\cdots+q^1y^1+\cdots\right)\left(\frac{\vartheta_3^{13}-\vartheta_4^{13}}{2}(\tau,0)\frac{\vartheta_3-\vartheta_4}{2}(\tau,z)+\frac{\vartheta_3^{13}+\vartheta_4^{13}}{2}(\tau,0)\frac{\vartheta_3+\vartheta_4}{2}(\tau,z)\right).$$

$$= \left(q^0 y^0 + \cdots + q^1 y^1 + \cdots\right)\left(26 q^1 (y^{+1} + y^{-1}) + \cdots + 1 + \cdots\right). \tag{5.103}$$

A similar analysis shows that there are one **26** and two **1** of SO(26).

Finally, we consider the R-sector terms of $\hat{F}^{\rm het}_{0,0}$:

$$\begin{aligned}
&\hat{F}^{(-)\rm R}_{0,0}(\tau, z) \cdot q^{\frac{1}{4} - \frac{k_{\rm min}+4}{8(k_{\rm min}+2)}} \\
&= q^{\frac{1}{4} - \frac{k_{\rm min}+4}{8(k_{\rm min}+2)}} \left(\chi^{0,-1}_{-1}(\tau, 0)\Theta_{k_{\rm min}+4, 2(k_{\rm min}+2)(k_{\rm min}+4)}\left(\tau, \frac{z}{k_{\rm min}+4}\right) + \cdots\right) \\
&\qquad\qquad \cdot \left(\Theta_{1,2}(\tau,0)\Theta_{-1,2}(\tau,z) + \Theta_{-1,2}(\tau,0)\Theta_{1,2}(\tau,z)\right). \\
&= \left(q^{\frac{1}{4}} y^{\frac{1}{2}} + \cdots\right)\left(q^{\frac{1}{4}}(y^{+\frac{1}{2}} + y^{-\frac{1}{2}}) + \cdots\right).
\end{aligned} \tag{5.104}$$

In the $E_8 \times E_8$ case, this is converted to

$$\to \left(q^{\frac{1}{4}} y^{\frac{1}{2}} + \cdots\right)\left(16 q^{\frac{3}{4}}(y^{+\frac{1}{2}} + y^{-\frac{1}{2}}) + \cdots\right). \tag{5.105}$$

Again, $q^{\frac{1}{4}} y^{\frac{1}{2}}$ is in the $n_{\rm cluster} = -\frac{1}{2}$ cluster and hence chooses only the first 16. In the SO(32) case,

$$\to \left(q^{\frac{1}{4}} y^{\frac{1}{2}} + \cdots\right)\left(2^{12} q^{\frac{7}{4}}(y^{+\frac{1}{2}} + y^{-\frac{1}{2}}) + \cdots\right), \tag{5.106}$$

and therefore no $h = 1$ states.

In all, $\hat{F}^{\rm het}_{0,0}$ has exactly the same set of $h = 1$ states as an $\hat{F}^{\rm het}_{j,j+2}$ does.

Massless spectrum for even $k_{\rm min}$

For the $A_{k_{\rm min}+1}$ modular invariant model with $k_{\rm min}$ odd, we have seen that there are $\frac{k_{\rm min}+3}{2}$ sets of massless $N = 1$ chiral multiplets, in the $\mathbf{10} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{16}$ representation of SO(10) for the $E_8 \times E_8$ theory, and in the $\mathbf{26} \oplus \mathbf{1} \oplus \mathbf{1}$ representation of SO(26) for the SO(32) theory. For the $A_{k_{\rm min}+1}$ modular invariant model with $k_{\rm min}$ even, we have similarly $\frac{k_{\rm min}}{2} + 1$ sets of massless $N = 1$ chiral multiplets in the same representations. In addition, there also exist non-localized "massless" matter fields corresponding to the continuous series representations that reach the boundary of the unitary region, as is the case in the type II spectrum.

6. Examples

6.1 Type II massless spectrum for $k_{\rm min} = 1$

The central charge of the $N = 2$ minimal model is $c_{\rm min} = 1$. The central charge of the $\mathrm{SL}(2, \mathbf{R})/\mathrm{U}(1)$ coset CFT is then $9 - 1 = 8$, and hence

$$\kappa = \frac{16}{5}, \quad k = \frac{6}{5}. \tag{6.1}$$

The string functions for $k_{\rm min} = 1$ are simply

$$c^l_m(\tau) = \frac{\delta^{(\rm mod 2)}_{m,l}}{\eta(\tau)}, \tag{6.2}$$

where $l = 0, 1$. The $k_{\rm min} = 1$ minimal characters are

$$\chi_m^{l,s}(\tau, z) = \frac{\delta_{m,l+s}^{(\rm mod2)}}{\eta(\tau)} \Theta_{2m-3s,6}\left(\tau, \frac{z}{3}\right). \tag{6.3}$$

To find the massless spectrum, it is convenient to use the formulas for $\hat{F}_{l,2r}$ (1) given in appendix. Since

$$\hat{F}_{1,2r} = \hat{F}_{0,5-2r}, \tag{6.4}$$

we only consider $l = 0$. Then r takes values in $\mathbf{Z}_5$. Setting $k_{\rm min} = 1$, we obtain

$$\hat{F}_{0,2r}^{(-)}(\tau, z) = \frac{1}{\eta(\tau)} \Theta_{-4r,5}\left(\tau, \frac{z}{5}\right) \frac{1}{2} \hat{\Lambda}_2(\tau, z), \tag{6.5}$$

$$\hat{F}_{0,2r}^{(+)}(\tau, z) = \frac{1}{\eta(\tau)} \Theta_{-4r+5,5}\left(\tau, \frac{z}{5}\right) \frac{1}{2} \hat{\Lambda}_1(\tau, z). \tag{6.6}$$

The NS- and R-sector spectra can be considered separately by writing

$$\hat{\Lambda}_1(\tau, z) = \hat{\Lambda}_1^{\rm NS}(\tau, z) - \hat{\Lambda}_1^{\rm R}(\tau, z), \tag{6.7}$$

$$\frac{1}{2}\hat{\Lambda}_1^{\rm NS}(\tau, z) \equiv \Theta_{1,1}(\tau, z)\Theta_{(0,0)}(\tau; 0, z), \tag{6.8}$$

$$\frac{1}{2}\hat{\Lambda}_1^{\rm R}(\tau, z) \equiv \Theta_{0,1}(\tau, z)\Theta_{(1,1)}(\tau; 0, z) \tag{6.9}$$

and

$$\hat{\Lambda}_2(\tau, z) = \hat{\Lambda}_2^{\rm NS}(\tau, z) - \hat{\Lambda}_2^{\rm R}(\tau, z), \tag{6.10}$$

$$\frac{1}{2}\hat{\Lambda}_2^{\rm NS}(\tau, z) \equiv \Theta_{0,1}(\tau, z)\Theta_{(0,2)}(\tau; 0, z), \tag{6.11}$$

$$\frac{1}{2}\hat{\Lambda}_2^{\rm R}(\tau, z) \equiv \Theta_{1,1}(\tau, z)\Theta_{(1,-1)}(\tau; 0, z), \tag{6.12}$$

where

$$\Theta_{(s,s')}(\tau; z, z') \equiv \sum_{\nu \in \mathbf{Z}_2} \Theta_{s+2\nu,2}(\tau, z)\Theta_{s'+2\nu,2}(\tau, z'). \tag{6.13}$$

We define $\hat{F}_{l,2r}^{(\pm)\rm NS}$ as formulas similar to (B.33), (B.35) but with $\hat{\Lambda}_2(\tau, z)$, $\hat{\Lambda}_1(\tau, z)$ being replaced with $\hat{\Lambda}_2^{\rm NS}(\tau, z)$, $\hat{\Lambda}_1^{\rm NS}(\tau, z)$, respectively, and

$$\hat{F}_{l,2r}^{\rm NS} \equiv \frac{1}{2}\left(\hat{F}_{l,2r}^{(-)\rm NS} + \hat{F}_{l,2r}^{(+)\rm NS}\right). \tag{6.14}$$

Among them, only $\hat{F}_{0,0}^{\rm NS}$, $\hat{F}_{0,-4}^{\rm NS}$ and $\hat{F}_{0,+2}^{\rm NS}(= \hat{F}_{0,-8}^{\rm NS})$ have theta functions whose U(1) charges are in the ranges (5.69).

$\hat{F}_{0,0}^{\rm NS}(\tau, z)$ has an expansion

$$2\hat{F}_{0,0}^{\rm NS}(\tau, z) = (\hat{F}_{0,0}^{(-)\rm NS} + \hat{F}_{0,0}^{(+)\rm NS})(\tau, z)$$

$$
\begin{aligned}
&= \frac{1}{\eta(\tau)} \underbrace{\left(1 + q(y + y^{-1}) + \cdots\right)}_{\Theta_{0,5}\left(\tau, \frac{z}{5}\right)\Theta_{0,1}(\tau,z)} \underbrace{\left(q^{\frac{1}{2}}(y + y^{-1} + 2) + \cdots\right)}_{\Theta_{(0,2)}(\tau;0,z)} \\
&\quad + \frac{1}{\eta(\tau)} \underbrace{\left(q(y^2 + y^{-2} + 2) + \cdots\right)}_{\Theta_{5,5}\left(\tau, \frac{z}{5}\right)\Theta_{1,1}(\tau,z)} \underbrace{\left(1 + 2q(y + y^{-1}) + \cdots\right)}_{\Theta_{(0,0)}(\tau;0,z)}.
\end{aligned}
\tag{6.15}
$$

The cluster number n_{cluster} can be read off from the power of y in the expansion of $\Theta_{*,5}(\tau, \frac{z}{5})\Theta_{*,1}(\tau, z)$; if the power satisfies

$$
\frac{1}{2} + n < (\text{The power}) < \frac{k+1}{2} + n \tag{6.16}
$$

for some $n \in \mathbf{Z}$ ($\in \mathbf{Z} + \frac{1}{2}$) for the NS (R) sector, then $n_{\text{cluster}} = n$.

The first line of (6.15) contains "1" in the first parenthesis; this is in the $n_{\text{cluster}} = -1$ cluster, for which $F^{(\text{NS})} \geq 1$.[14] Therefore, it can be paired with $q^{\frac{1}{2}}y$ in the second parenthesis, but not with $q^{\frac{1}{2}}y^{-1}$. The second line has no $q^{\frac{1}{2}}$ terms. Thus we found a single $h = \frac{1}{2}$ state in $\hat{F}^{\text{NS}}_{0,0}(\tau, z)$.

Also $\hat{F}^{\text{NS}}_{0,2}(\tau, z)$ is expanded as

$$
\begin{aligned}
2\hat{F}^{\text{NS}}_{0,2}(\tau, z) \times q^{\frac{1}{5}} &= q^{\frac{1}{5}}(\hat{F}^{(-)\text{NS}}_{0,2} + \hat{F}^{(+)\text{NS}}_{0,2})(\tau, z) \\
&= \frac{q^{\frac{1}{5}}}{\eta(\tau)} \underbrace{\left(q^{\frac{4}{5}}y^{-\frac{2}{5}} + \cdots \right)}_{\Theta_{-4,5}\left(\tau, \frac{z}{5}\right)\Theta_{0,1}(\tau,z)} \underbrace{\left(q^{\frac{1}{2}}(y + y^{-1} + 2) + \cdots\right)}_{\Theta_{(0,2)}(\tau;0,z)} \\
&\quad + \frac{q^{\frac{1}{5}}}{\eta(\tau)} \underbrace{\left(q^{\frac{1}{20}+\frac{1}{4}}y^{\frac{1}{10}}(y^{\frac{1}{2}} + y^{-\frac{1}{2}}) + \cdots\right)}_{\Theta_{+1,5}\left(\tau, \frac{z}{5}\right)\Theta_{1,1}(\tau,z)} \underbrace{\left(1 + 2q(y + y^{-1}) + \cdots\right)}_{\Theta_{(0,0)}(\tau;0,z)},
\end{aligned}
\tag{6.17}
$$

where we have included the factor[15] of $q^{\frac{1}{5}}$ coming from the extra weight

$$
\frac{1}{4} - \frac{c_{\text{min}}}{24} = \frac{5}{24} \tag{6.18}
$$

because of the shortage of the eta functions in the denominator of the partition function, minus the "drop" coming from the imaginary momentum of a possible discrete state

$$
\frac{1}{k}\left(J_0^{\text{tot}} + \frac{1}{2} + \frac{k}{2}\right)^2 = \frac{k_{\text{min}} + 4}{2(k_{\text{min}} + 2)}\left(\frac{J_0^{\text{U}(1)}}{k_{\text{min}} + 4} - \frac{1}{2} - n_{\text{cluster}}\right)^2 = \frac{1}{120} \tag{6.19}
$$

($\frac{5}{24} - \frac{1}{120} = \frac{1}{5}$). In this case the term $q^{\frac{1}{5}+\frac{1}{20}+\frac{1}{4}}y^{\frac{1}{10}+\frac{1}{2}} = q^{\frac{1}{2}}y^{\frac{3}{5}}$ in the last line indicates a massless discrete state in the $n_{\text{cluster}} = 0$ cluster, while the other $q^{\frac{1}{2}}y^{\frac{1}{10}-\frac{1}{2}}$ is in the $n_{\text{cluster}} = -1$ cluster, for which $F^{(\text{NS})}$ must be ≥ 1, and does not corresponds to any discrete states.

[14] To be sure, $F^{(\text{NS})}$ here is the fermion number in the NS sector defined in section?, which should not be confused with $\hat{F}^{\text{NS}}_{l,2r}(\tau, z)$.

[15] For $\hat{F}^{\text{NS}}_{0,0}$ such a factor is absent because in that case the imaginary momentum factor precisely cancels the extra conformal weight of the continuous series.

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We can similarly expand $q^{\frac{4}{30}}\hat{F}^{\mathrm{NS}}_{0,-4}(\tau,z)$ (where $\frac{4}{30}$ is again $\frac{5}{24}$ minus the imaginary momentum contribution $\frac{3}{40}$), but can find no $h=\frac{1}{2}$ states.

Therefore, for the A_2 modular invariant (in which $k_{\min}=1$), we find two massless states in the NS-NS sector, one from $\hat{F}^{\mathrm{NS}}_{0,0}(\hat{F}^{\mathrm{NS}}_{0,0})^*$ and the other from $\hat{F}^{\mathrm{NS}}_{0,2}(\hat{F}^{\mathrm{NS}}_{0,2})^*$. Due to (6.4), $\hat{F}^{\mathrm{NS}}_{1,5}(\hat{F}^{\mathrm{NS}}_{1,5})^*$ and $\hat{F}^{\mathrm{NS}}_{1,7}(\hat{F}^{\mathrm{NS}}_{1,7})^*$ are also in the summation (5.66).

We study the Ramond sector in a similar way, and find as many $h=\frac{1}{2}$ states as in the NS sector due to the supersymmetry. All in all, they are two vector multiplets for the typeIIA case and two hypermultiplets for the typeIIB case, as we discussed in section 5.5.

6.2 $E_8\times E_8$ heterotic massless spectrum for $k_{\min}=1$

Next we turn to the $E_8\times E_8$ heterotic string compactification. Again, we set $k_{\min}=1$. As we did for $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ in the last subsection, we write

$$\hat{\Lambda}_1^{E_8\times E_8}(\tau,z) = \hat{\Lambda}_1^{E_8\times E_8,\mathrm{NS}}(\tau,z)+\hat{\Lambda}_1^{E_8\times E_8,R}(\tau,z), \tag{6.20}$$

$$\frac{\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,\mathrm{NS}}(\tau,z)}{\eta^{14}(\tau)} \equiv \Theta_{1,1}(\tau,z)\Big(B_v^{(10)}(\tau,0)B_0^{(2)}(\tau,z)+B_0^{(10)}(\tau,0)B_v^{(2)}(\tau,z)\Big)B^{(E_8)}(\tau,0), \tag{6.21}$$

$$\frac{\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,R}(\tau,z)}{\eta^{14}(\tau)} \equiv \Theta_{0,1}(\tau,z)\left(B_s^{(10)}(\tau,0)B_s^{(2)}(\tau,z)+B_{\bar{s}}^{(10)}(\tau,0)B_{\bar{s}}^{(2)}(\tau,z)\right)B^{(E_8)}(\tau,0), \tag{6.22}$$

and

$$\hat{\Lambda}_2^{E_8\times E_8}(\tau,z) = \hat{\Lambda}_2^{E_8\times E_8,\mathrm{NS}}(\tau,z)+\hat{\Lambda}_2^{E_8\times E_8,R}(\tau,z), \tag{6.23}$$

$$\frac{\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,\mathrm{NS}}(\tau,z)}{\eta^{14}(\tau)} \equiv \Theta_{0,1}(\tau,z)\left(B_0^{(10)}(\tau,0)B_0^{(2)}(\tau,z)+B_v^{(10)}(\tau,0)B_v^{(2)}(\tau,z)\right)B^{(E_8)}(\tau,0), \tag{6.24}$$

$$\frac{\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,R}(\tau,z)}{\eta^{14}(\tau)} \equiv \Theta_{1,1}(\tau,z)\left(B_s^{(10)}(\tau,0)B_s^{(2)}(\tau,z)+B_{\bar{s}}^{(10)}(\tau,0)B_{\bar{s}}^{(2)}(\tau,z)\right)B^{(E_8)}(\tau,0). \tag{6.25}$$

Then

$$\hat{F}_{l,2r}^{E_8\times E_8}(\tau,z) = \left(\hat{F}_{l,2r}^{E_8\times E_8,\mathrm{NS}}+\hat{F}_{l,2r}^{E_8\times E_8,R}\right)(\tau,z). \tag{6.26}$$

Let us first consider $2\hat{F}_{0,0}^{E_8\times E_8}$:

$$\begin{aligned}
2\hat{F}_{0,0}^{E_8\times E_8,\mathrm{NS}}(\tau,z) &= \left(\hat{F}_{0,0}^{E_8\times E_8,(-)\mathrm{NS}}+\hat{F}_{0,0}^{E_8\times E_8,(+)\mathrm{NS}}\right)(\tau,z),\\
&= \frac{1}{\eta(\tau)}\left(\Theta_{0,5}\left(\tau,\frac{z}{5}\right)\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,\mathrm{NS}}(\tau,z)+\Theta_{5,5}\left(\tau,\frac{z}{5}\right)\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,\mathrm{NS}}(\tau,z)\right)\\
&= \frac{1}{\eta(\tau)}\underbrace{(1+q(y+y^{-1})+\cdots)}_{\Theta_{0,5}\left(\tau,\frac{z}{5}\right)\Theta_{0,1}(\tau,z)}\underbrace{(\,1+q(10y+10y^{-1}+40)+\cdots\,)}_{\text{Fermion theta fns. of }\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,\mathrm{NS}}(\tau,z)}
\end{aligned}$$

$$+\frac{1}{\eta(\tau)}(\underbrace{q(y^2+y^{-2}+2)+\cdots}_{\Theta_{5,5}(\tau,\frac{z}{5})\Theta_{1,1}(\tau,z)})(\underbrace{q^{\frac{1}{2}}(y+y^{-1}+10)+\cdots}_{\text{Fermion theta fns. of }\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,\mathrm{NS}}(\tau,z)}), \tag{6.27}$$

$$\begin{aligned}
2\hat{F}_{0,0}^{E_8\times E_8,R}(\tau,z) &= \left(\hat{F}_{0,0}^{E_8\times E_8,(-)R}+\hat{F}_{0,0}^{E_8\times E_8,(+)R}\right)(\tau,z),\\
&= \frac{1}{\eta(\tau)}\left(\Theta_{0,5}\left(\tau,\frac{z}{5}\right)\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,R}(\tau,z)+\Theta_{5,5}\left(\tau,\frac{z}{5}\right)\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,R}(\tau,z)\right)\\
&= \frac{1}{\eta(\tau)}(\underbrace{q^{\frac{1}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots}_{\Theta_{0,5}(\tau,\frac{z}{5})\Theta_{1,1}(\tau,z)})(\underbrace{16q^{\frac{3}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots}_{\text{Fermion theta fns. of }\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,R}(\tau,z)})\\
&\quad +\frac{1}{\eta(\tau)}(\underbrace{q^{\frac{5}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots}_{\Theta_{5,5}(\tau,\frac{z}{5})\Theta_{0,1}(\tau,z)})(\underbrace{16q^{\frac{3}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots}_{\text{Fermion theta fns. of }\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,R}(\tau,z)}).
\end{aligned}\tag{6.28}$$

In this case we look for $h=1$ states. Again, we can know which cluster the spectrum belongs to by the power of y in the expansion of $\Theta_{*,5}(\tau,\frac{z}{5})\Theta_{*,1}(\tau,z)$ because they arose from the composition of the U(1) theta function (and the minimal $N=2$ theta function which has no J_0^{tot} charge).

In the first line of (6.27), the term "1" in the first parenthesis belongs to the $n_{\mathrm{cluster}}=-1$ cluster. Therefore, with a lowest $L_0^{\mathrm{SU}(2,\mathbf{R})}(=0)$ state, $F^{(\mathrm{NS})}$ must be ≥ 1 and it can be paired with $10qy$, but not with $10qy^{-1}$. This gives a single **10** representation of SO(10). As we noted in section 5.7, if $J_{-1}^{+}|0,-\frac{\kappa}{2}\rangle$ (rather than $|0,-\frac{\kappa}{2}\rangle$) is chosen as the state in the SL$(2,\mathbf{R})$ module $\mathcal{H}_{-,(0,-\frac{\kappa}{2})}^{\mathrm{SL}(2,\mathbf{R})}$, then $F^{(\mathrm{NS})}$ is relaxed to ≥ 0 and the two "1"s can be paired. This gives a singlet.

The "qy" term is in the $n_{\mathrm{cluster}}=0$ cluster and can be paired with 1 in the second parenthesis. This is another singlet. On the other hand, the "qy^{-1}" term is in the $n_{\mathrm{cluster}}=-2$ cluster and $F^{(\mathrm{NS})}$ must be ≥ 2. Therefore, it does not give rise to any $h=1$ states. Also, no $h=1$ states arise from the second line of (6.27). A similar analysis can be made for the Ramond sector (6.28). This confirms sixteen $h=1$ states from the first line. In all, we find a set of $10+1+1+16=28$, $h=1$ states in $2\hat{F}_{0,0}^{E_8\times E_8}$.

Next we consider $2\hat{F}_{0,2}^{E_8\times E_8}$:

$$\begin{aligned}
2\hat{F}_{0,2}^{E_8\times E_8,\mathrm{NS}}(\tau,z)\times q^{\frac{1}{5}} &= \left(\hat{F}_{0,2}^{E_8\times E_8,(-)\mathrm{NS}}+\hat{F}_{0,2}^{E_8\times E_8,(+)\mathrm{NS}}\right)(\tau,z)\times q^{\frac{1}{5}}\\
&= \frac{q^{\frac{1}{5}}}{\eta(\tau)}\left(\Theta_{-4,5}\left(\tau,\frac{z}{5}\right)\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,\mathrm{NS}}(\tau,z)+\Theta_{1,5}\left(\tau,\frac{z}{5}\right)\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,\mathrm{NS}}(\tau,z)\right)\\
&= \frac{q^{\frac{1}{5}}}{\eta(\tau)}(\underbrace{q^{\frac{4}{5}}y^{-\frac{2}{5}}+\cdots}_{\Theta_{-4,5}(\tau,\frac{z}{5})\Theta_{0,1}(\tau,z)})(\underbrace{1+q(10y+10y^{-1}+40)+\cdots}_{\text{Fermion theta fns. of }\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,\mathrm{NS}}(\tau,z)})\\
&\quad +\frac{q^{\frac{1}{5}}}{\eta(\tau)}(\underbrace{q^{\frac{3}{10}}(y^{\frac{3}{5}}+y^{-\frac{2}{5}})+\cdots}_{\Theta_{1,5}(\tau,\frac{z}{5})\Theta_{1,1}(\tau,z)})(\underbrace{q^{\frac{1}{2}}(y+y^{-1}+10)+\cdots}_{\text{Fermion theta fns. of }\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,\mathrm{NS}}(\tau,z)}),
\end{aligned}\tag{6.29}$$

$$2\hat{F}_{0,2}^{E_8\times E_8,R}(\tau,z)\times q^{\frac{1}{5}} = \left(\hat{F}_{0,2}^{E_8\times E_8,(-)R}+\hat{F}_{0,2}^{E_8\times E_8,(+)R}\right)(\tau,z)\times q^{\frac{1}{5}}$$

$$
\begin{aligned}
&= \frac{q^{\frac{1}{5}}}{\eta(\tau)}\Big(\Theta_{0,5}\Big(\tau,\frac{z}{5}\Big)\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,R}(\tau,z)+\Theta_{5,5}\Big(\tau,\frac{z}{5}\Big)\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,R}(\tau,z)\Big)\\
&= \frac{q^{\frac{1}{5}}}{\eta(\tau)}(\underbrace{q^{\frac{21}{20}}(y^{\frac{1}{10}}+y^{-\frac{9}{10}})+\cdots}_{\Theta_{-4,5}(\tau,\frac{z}{5})\Theta_{1,1}(\tau,z)})(\underbrace{16q^{\frac{3}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots}_{\text{Fermion theta fns. of }\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8,R}(\tau,z)})\\
&\quad+\frac{q^{\frac{1}{5}}}{\eta(\tau)}(\underbrace{q^{\frac{1}{20}}y^{\frac{1}{10}}+\cdots}_{\Theta_{1,5}(\tau,\frac{z}{5})\Theta_{0,1}(\tau,z)})(\underbrace{16q^{\frac{3}{4}}(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+\cdots}_{\text{Fermion theta fns. of }\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8,R}(\tau,z)}). \qquad (6.30)
\end{aligned}
$$

The first line of the NS-sector expansion (6.29) contains one q^1 term ($= q^{\frac{1}{5}+\frac{4}{5}}y^{-\frac{2}{5}}$), but it comes from the $n_{\text{cluster}} = -1$ cluster for which $F^{\text{NS}} \geq 1$ (that is, "1" in the second parenthesis cannot be paired), and hence does not give rise to a discrete state. The second line has terms proportional to $q^{\frac{1}{5}+\frac{3}{10}+\frac{1}{2}} = q^1$:

$$
q\left(y^{\frac{3}{5}}+y^{-\frac{2}{5}}\right)\left(10+y+y^{-1}\right). \qquad (6.31)
$$

$y^{\frac{3}{5}}$ is in the $n_{\text{cluster}} = 0$ cluster, while $y^{-\frac{2}{5}}$ the $n_{\text{cluster}} = -1$ cluster. Therefore, due to the constraint, only $qy^{\frac{3}{5}}\cdot 10$, $qy^{\frac{3}{5}}\cdot y$ and $qy^{-\frac{2}{5}}\cdot y$ correspond to discrete states. The first is in the **10** representation, while the latter two are singlets. The R-sector expansion (6.30) similarly gives rise to a **16** of SO(10) from the second line of (6.30).

Since the left- and right-moving $\hat{F}_{l,2r}$'s with the same r are paired in (5.66), and since we have seen that $(\hat{F}_{0,-4})^*$ has no $h = \frac{1}{2}$ states, we do not need to consider $\hat{F}_{0,-4}^{E_8\times E_8}$.

To summarize the $k_{\min} = 1$ $E_8 \times E_8$ heterotic massless spectrum, we have found two sets of $N = 1$ scalar multiplets in $\mathbf{10}\oplus\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{16}$ of SO(10).

6.3 The three generation model

Let us consider the $k_{\min} = 3$, A_4 modular invariant model of the $E_8 \times E_8$ heterotic string theory. According to the rule we have found in section 5.7, there appear three generations of massless matter multiplets in $\mathbf{10}\oplus\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{16}$ of SO(10), or $\mathbf{27}\oplus\mathbf{1}$ of E_6. They are localized on a four-dimensional spacetime. It is interesting to note that these three generations are *not* on an equal footing; for example, one generation has the **10** representation from the $n_{\text{cluster}} = -1$ cluster, whereas in the other two generations it comes from the $n_{\text{cluster}} = 0$ cluster. In a more realistic phenomenological application, this fact might be used as the origin of the differences among the generations observed in Nature.

There are no localized gauge fields. Gauge fields correspond to the continuous series representations of SL(2, $\mathbf{R}$) and acquire a mass from the Liouville energy. They propagate into the bulk, as is the case for the graviton. This situation is analogous to the local GUT in the standard orbifold compactification, where the matter fields in the twisted sector constitute locally a representation of a possibly larger group than the actual unbroken gauge symmetry.

7. Localized modes in six dimensions

In this section we generalize the analysis in the previous sections to six dimensions. The partition function for the internal Calabi-Yau is the same except the difference of the level k, and it has been shown [13] that the discrete spectrum of it for the ALE spaces correctly reflects their topological data.

As we did in four dimensions, we couple the internal part to a free superconformal field theory describing the six-dimensional flat Minkowski space, perform a GSO projection in a suitable way before the continuous and discrete representations are separated. This is good because, as we mentioned in Introduction, the couplings between the discrete states and the CFT for the Minkowski space are automatically consistent with the modular invariance. A state corresponding to a discrete series representation is always associated with some continuous spectrum of states. They arise from the same integral with different contours. Therefore, the couplings between the discrete states and the Minkowski CFT are not arbitrary but constrained by modular invariance of the continuous sector.

In the Calabi-Yau twofold case, the relation between the levels of the $\mathrm{SL}(2,\mathbf{R})$ WZW and the $N = 2$ minimal modes is

$$\frac{3\kappa}{\kappa-2}+\frac{3k_{\rm min}}{k_{\rm min}+2}=6, \tag{7.1}$$

and hence

$$\begin{aligned}\kappa-2 &= k_{\rm min}+2\\ &\equiv k.\end{aligned} \tag{7.2}$$

Unlike in the threefold case, k is always an integer for a non-negative integer $k_{\rm min}$.

We again consider the internal CFT partition function

$$\begin{aligned}Z^{(\nu)}_{\rm CY}(\tau) &= C\int_0^1 ds_1\int_0^1 ds_2\frac{|\Theta_{\nu,2}(\tau,s_1\tau-s_2)|^2}{|\vartheta_1(\tau,s_1\tau-s_2)|^2}\\ &\quad\cdot\sqrt{\frac{\tau_2}{k}}\sum_{m,\tilde m}e^{-k\pi\tau_2 s_1^2}q^{\frac{m^2}{k}}e^{-2\pi i m(s_1\tau-s_2)}\bar q^{\frac{\tilde m^2}{k}}e^{+2\pi i\tilde m(s_1\bar\tau-s_2)}\end{aligned} \tag{7.3}$$

for $\nu\in\mathbf{Z}_4$, where $m=\frac{n-kw}{2}$, $\tilde m=-\frac{n+kw}{2}$ and $n,w\in\mathbf{Z}$. This is the same as (4.1) with a slight change of notation, and the Poisson resummation (4.2) has already been done. We set

$$\begin{aligned}m &\equiv kj+\frac{r}{2},\\ \tilde m &\equiv k\tilde j+\frac{\tilde r}{2}.\end{aligned} \tag{7.4}$$

Since both n and w are integers, j and $\tilde j$ run independently over $\mathbf{Z}$, whereas r and $\tilde r$ take values in $\mathbf{Z}_{2k}$ with a constraint $r+\tilde r=0 \bmod 2k$. Using this change of variables, $Z^{(\nu)}_{\rm CY}(\tau)$ can be put in the form

$$\begin{aligned}Z^{(\nu)}_{\rm CY}(\tau) &= C\int_0^1 ds_1\int_0^1 ds_2\frac{|\Theta_{\nu,2}(\tau,s_1\tau-s_2)|^2}{|\vartheta_1(\tau,s_1\tau-s_2)|^2}\\ &\quad\cdot\sqrt{\frac{\tau_2}{k}}e^{-k\pi\tau_2 s_1^2}\sum_{r,\tilde r}\Theta_{r,k}(\tau,s_1\tau-s_2)(\Theta_{\tilde r,k}(\tau,s_1\tau-s_2))^*.\end{aligned} \tag{7.5}$$

Since $k = k_{\min}+2$, these theta functions are the ones appearing in $F_l(\tau, z)$ (3.22), which was introduced in [13] to construct modular invariants for the ADE singularities. Therefore, generalizing the result in the previous sections, we can easily arrive at the supersymmetric modular invariant partition function including the discrete series contributions

$$Z_{\mathcal{M}_6\times CY(X_n)}(\tau) = C\int_0^1 ds_1\int_0^1 ds_2\sqrt{\frac{\tau_2}{k}}(q\bar{q})^{\frac{ks_1^2}{4}}\sum_{l,\tilde{l}}N_{l\tilde{l}}\frac{\hat{F}_l(\tau,s_1\tau-s_2)(\hat{F}_{\tilde{l}}(\tau,s_1\tau-s_2))^*}{|\eta^2(\tau)\vartheta_1(\tau,s_1\tau-s_2)|^2}, \tag{7.6}$$

where we have introduced a new set of functions

$$\begin{aligned}\hat{F}_l(\tau,z) &\equiv \frac{1}{2}\chi_l^{(k_{\min})}(\tau,0)\left(\vartheta_3^4-\vartheta_4^4-\vartheta_2^4+\tilde{\vartheta}_1^4\right)(\tau,z)\\ &= \sum_{\nu\in\mathbf{Z}_4}(-1)^\nu\sum_{m\in\mathbf{Z}_{2(k_{\min}+2)}}\chi_m^{l,\nu}(\tau,0)\sum_{\substack{\nu_0,\nu_1,\nu_2\in\mathbf{Z}_2\\ \nu_0+\nu_1+\nu_2\\ \equiv 1(\mathrm{mod}2)}}\Theta_{2\nu_0+\nu,2}(\tau,0)\Theta_{2\nu_1+\nu,2}(\tau,0)\\ &\qquad\cdot\Theta_{2\nu_2+\nu,2}(\tau,z)\Theta_{m,k_{\min}+2}(\tau,z)\end{aligned} \tag{7.7}$$

for $l = 0,\ldots,k_{\min}$. The z-dependences are so chosen that they match those of (7.5). Again, in going from $Z_{\mathrm{CY}}^{(\nu)}(\tau)$ to $Z_{\mathcal{M}_6\times CY(X_n)}(\tau)$, we have relaxed the constraint on r and $\tilde{r}$ in order to obtain a supersymmetric partition function. As before, we can show that $Z_{\mathcal{M}_6\times CY(X_n)}(\tau)$ is invariant under both the modular S- and T-transformations.

In order to separate the discrete series spectrum we define

$$\mathcal{H}_{F_l}^{(\nu)} \equiv \bigoplus_{m\in\mathbf{Z}_{2(k_{\min}+2)}}\bigoplus_{\substack{\nu_0,\nu_1,\nu_2\in Z_2\\ \nu_0+\nu_1+\nu_2\\ \equiv 1(\mathrm{mod}2)}}\left(\mathcal{H}_m^{(k_{\min})l,\nu}\otimes\mathcal{H}_{2\nu_0+\nu,2}\otimes\mathcal{H}_{2\nu_1+\nu,2}\otimes\mathcal{H}_{2\nu_2+\nu,2}\otimes\mathcal{H}_{m,k_{\min}+2}\right), \tag{7.8}$$

where various component modules are defined in section 5.1. Using this, we can express $\hat{F}_l(\tau,z)$ as

$$\begin{aligned}\hat{F}_l(\tau,z) =\, & i^{-1}\eta^4(\tau)\vartheta_1(\tau,z)\sum_{\nu\in\mathbf{Z}_4}(-1)^\nu\mathrm{Tr}_{\mathcal{H}_{+,(0,0)}^{\mathrm{SL}(2,\mathbf{R})}\otimes\mathcal{H}_{F_l}^{(\nu)}}q^{L_0^{\mathrm{SL}(2,\mathbf{R})}+L_0^{N=2}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{\mathrm{U}(1)}-\frac{c_{\min}+7}{24}}\\ &\cdot y^{J_0^3+F^{(\nu)}+J_0^{\mathrm{U}(1)}+\frac{1}{2}},\end{aligned} \tag{7.9}$$

and $Z_{\mathcal{M}_6\times CY(X_n)}(\tau)$ can be written in this case as

$$\begin{aligned}Z_{\mathcal{M}_6\times CY(X_n)}(\tau) =\, & C\sum_{l,\tilde{l}}N_{l,\tilde{l}}\sqrt{\frac{\tau_2}{k}}\int_0^1 ds_1\int_0^1 ds_2\\ &\cdot\sum_{\nu,\tilde{\nu}\in Z_4}(-1)^{\nu+\tilde{\nu}}\mathrm{Tr}_{\left(\mathcal{H}_{+,(0,0)}^{\mathrm{SL}(2,\mathbf{R})}\otimes\mathcal{H}_{F_l}^{(\nu)}\right)\otimes\left(\mathcal{H}_{+,(0,0)}^{\mathrm{SL}(2,\mathbf{R})}\otimes\mathcal{H}_{F_{\tilde{l}}}^{(\tilde{\nu})}\right)}\\ &\cdot q^{\frac{ks_1^2}{4}+L_0^{\mathrm{SL}(2,\mathbf{R})}+L_0^{N=2}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{(\nu)}+L_0^{\mathrm{U}(1)}-\frac{c_{\min}+7}{24}+s_1(J_0^3+F^{(\nu)}+J_0^{\mathrm{U}(1)}+\frac{1}{2})}\\ &\cdot\bar{q}^{\frac{ks_1^2}{4}+\tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})}+\tilde{L}_0^{N=2}+\tilde{L}_0^{(\nu)}+\tilde{L}_0^{(\nu)}+\tilde{L}_0^{(\nu)}+\tilde{L}_0^{\mathrm{U}(1)}-\frac{c_{\min}+7}{24}+s_1(\tilde{J}_0^3+\tilde{F}^{(\tilde{\nu})}+\tilde{J}_0^{\mathrm{U}(1)}+\frac{1}{2})}\end{aligned}$$

$$
\begin{aligned}
&\cdot\ e^{-2\pi i s_2 (J_0^3 + F^{(\nu)} + J_0^{\mathrm{U}(1)} - \tilde{J}_0^3 - \tilde{F}^{(\tilde{\nu})} - \tilde{J}_0^{\mathrm{U}(1)})} \\
&\cdot\ \left|\eta^2(\tau\right|^2 .
\end{aligned} \tag{7.10}
$$

As before, the s_2 integration yields the constraint

$$
J_0^3 + F^{(\nu)} + J_0^{\mathrm{U}(1)} = \tilde{J}_0^3 + \tilde{F}^{(\tilde{\nu})} + \tilde{J}_0^{\mathrm{U}(1)}, \tag{7.11}
$$

the left and right hand sides of which we call J_0^{tot} and $\tilde{J}_0^{\mathrm{tot}}$, respectively.

After a similar Fourier transformation and a spectral flow operation, we obtain

$$
\begin{aligned}
Z_{\mathcal{M}_6 \times CY(X_n)}(\tau) = C \sum_{l,\tilde{l}} N_{l,\tilde{l}} \frac{|\eta^2(\tau)|^2}{-2\pi k} &\sum_{\nu,\tilde{\nu}\in \mathbf{Z}_4} (-1)^{\nu+\tilde{\nu}} \\
&\cdot \Bigg(\mathrm{Tr}_{\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})} \otimes \mathcal{H}^{(\nu)}_{F_l}\right) \otimes \left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{-,(0,-\frac{\kappa}{2})} \otimes \mathcal{H}^{(\tilde{\nu})}_{F_{\tilde{l}}}\right)} \\
&\int_{-\infty}^{\infty} \frac{dp}{ip + J_0^{\mathrm{tot}} + \frac{1}{2}} q^{\frac{1}{k}\left(p+\frac{ik}{2}\right)^2 + L_0^{\mathrm{SL}(2,\mathbf{R})} + L_0^{N=2} + L_0^{(\nu)} + L_0^{(\nu)} + L_0^{(\nu)} + L_0^{\mathrm{U}(1)} - \frac{c_{\mathrm{min}}+7}{24}} \\
&\qquad \cdot\ \bar{q}^{\frac{1}{k}\left(p+\frac{ik}{2}\right)^2 + \tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})} + \tilde{L}_0^{N=2} + \tilde{L}_0^{(\nu)} + \tilde{L}_0^{(\nu)} + \tilde{L}_0^{(\nu)} + \tilde{L}_0^{\mathrm{U}(1)} - \frac{c_{\mathrm{min}}+7}{24}} \\
&- \mathrm{Tr}_{\left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)} \otimes \mathcal{H}^{(\nu)}_{F_l}\right) \otimes \left(\mathcal{H}^{\mathrm{SL}(2,\mathbf{R})}_{+,(0,0)} \otimes \mathcal{H}^{(\tilde{\nu})}_{F_{\tilde{l}}}\right)} \\
&\quad \int_{-\infty}^{\infty} \frac{dp}{ip + J_0^{\mathrm{tot}} + \frac{1}{2}} q^{\frac{p^2}{k} + L_0^{\mathrm{SL}(2,\mathbf{R})} + L_0^{N=2} + L_0^{(\nu)} + L_0^{(\nu)} + L_0^{(\nu)} + L_0^{\mathrm{U}(1)} - \frac{c_{\mathrm{min}}+7}{24}} \\
&\qquad \cdot\ \bar{q}^{\frac{p^2}{k} + \tilde{L}_0^{\mathrm{SL}(2,\mathbf{R})} + \tilde{L}_0^{N=2} + \tilde{L}_0^{(\nu)} + \tilde{L}_0^{(\nu)} + \tilde{L}_0^{(\nu)} + \tilde{L}_0^{\mathrm{U}(1)} - \frac{c_{\mathrm{min}}+7}{24}} \Bigg) \Bigg|_{J_0^{\mathrm{tot}} = \tilde{J}_0^{\mathrm{tot}}}
\end{aligned} \tag{7.12}
$$

We now deform the contour of the first trace. There is a difference here. In the present case we have $k = k_{\mathrm{min}} + 2$, so that k grows linearly as k_{min}. Therefore, if we change the contour of p from $\mathbf{R}$ to $\mathbf{R} + \frac{ik}{2}$, then it sweeps across many ($\left[\frac{k}{2}\right]$ at most) pole singularities through the deformation. (In contrast, k does not exceed two in the threefold case considered in the preceding sections, and therefore the contour picks up at most a single pole contribution.)

As before, the partition function gets pole contributions from the states having

$$
-\frac{k_{\mathrm{min}} + 3}{2} < J_0^{\mathrm{tot}} (= \tilde{J}_0^{\mathrm{tot}}) < \frac{1}{2}. \tag{7.13}
$$

To see the massless spectrum, what we need to do is to find conformal weight $\frac{1}{2}$ NS-sector states that satisfy (7.13) and $J_0^{\mathrm{tot}} = \tilde{J}_0^{\mathrm{tot}}$ in

$$
\sum_{l,\tilde{l}} N_{l,\tilde{l}} \left| y^{\frac{1-\kappa}{2}} \right|^2 \frac{\hat{F}_l(\tau,z) (\hat{F}_{\tilde{l}}(\tau,z))^*}{\left| \tilde{\vartheta}_1(\tau,z) \eta(\tau) \right|^2} \tag{7.14}
$$

with taking into account the Liouville energy (the shortage of eta functions) and the drop of weight due to the imaginary momentum of the discrete states. C has been chosen to be

k in this six-dimensional case. The R-sector states follow from supersymmetry. (7.14) is an analogue of (5.66) and can be similarly derived.

Repeating the previous steps, we set

$$J_0^3 \equiv -\frac{\kappa}{2} - N \qquad (N = 0, 1, 2, \ldots). \tag{7.15}$$

Then

$$\begin{aligned} J_0^{\rm tot} &= -\frac{\kappa}{2} - N + F^{(\nu)} + J_0^{\rm U(1)} \\ &\equiv -\frac{\kappa}{2} - n_{\rm cluster} + J_0^{\rm U(1)}. \end{aligned} \tag{7.16}$$

where, again, we defined the number

$$n_{\rm cluster} = N - F^{(\nu)} \tag{7.17}$$

($\in \mathbf{Z}$ for the NS sector) to label different allowed ranges of $J_0^{\rm U(1)}$. Using this number, we have

$$n_{\rm cluster} + \frac{1}{2} \;<\; J_0^{\rm U(1)} \;<\; n_{\rm cluster} + \frac{k_{\rm min}+3}{2}. \tag{7.18}$$

Unlike the threefold case, these ranges overlap with the neighboring ones. The imaginary momentum factor is (for the holomorphic part)

$$q^{-\frac{1}{k_{\rm min}+2}\left(J_0^{\rm U(1)} - \frac{1}{2} - n_{\rm cluster}\right)^2}. \tag{7.19}$$

Note that due to (7.15) and (7.17) a discrete state must satisfy $F^{(\nu)} \geq -n_{\rm cluster}$, the fact already used extensively in the threefold analysis.

Let us find weight $\frac{1}{2}$ state contributions to the NS-sector ($\nu = 0, 2$) terms of $\hat{F}_l(\tau, z)$ (7.7).

If $\nu = 0$, at least one of the fermion theta must be $\Theta_{2,2}$, and $\chi_m^{l,0}$ is (anti-)chiral primary for $m = \pm l$.

If $m = +l$, we see from (7.19) that, among several choices of $n_{\rm cluster}$, a lower $n_{\rm cluster}$ results in a larger drop of conformal weight. On the other hand, if $n_{\rm cluster}$ is negatively large, $F^{(\nu)} \geq -n_{\rm cluster}$ means that the Liouville fermion number $F^{(\nu)}$ is also large. It turns out that $n_{\rm cluster} = -1$ gives the lowest value of conformal weight. In this case, the power of q is

$$\underbrace{\frac{1}{8}}_{\text{Liouville energy}} + \underbrace{\frac{l}{2(k_{\rm min}+2)} - \frac{k_{\rm min}}{8(k_{\rm min}+2)}}_{\chi_m^{l,\nu}} + \underbrace{\frac{l^2}{4(k_{\rm min}+2)}}_{\Theta_{m,k_{\rm min}+2}} + \underbrace{\frac{1}{2}}_{\Theta_{2,2}} \underbrace{-\frac{(l+1)^2}{4(k_{\rm min}+2)}}_{\text{imaginary momentum factor}} = \frac{1}{2}. \tag{7.20}$$

If $m = -l$, then $n_{\rm cluster} < -1$ and it does not give any weight $\frac{1}{2}$ states.

Next we consider $\nu = 2$. In this case all the fermion thetas can be $\Theta_{0,2}$ simultaneously. $\chi_m^{l,2} = \chi_{m+k_{\rm min}+2}^{k_{\rm min}-l,0}$ is (anti-)chiral primary for $m = \pm(l+2)$.

If $m = l+2$, then the lowest conformal weight arises from $n_{\text{cluster}} = 0$. The counting of the various contributions is

$$\underbrace{\frac{1}{8}}_{\text{Liouville energy}} + \underbrace{\frac{k_{\min}-l}{2(k_{\min}+2)} - \frac{k_{\min}}{8(k_{\min}+2)}}_{\chi_m^{l,\nu}} + \underbrace{\frac{(l+2)^2}{4(k_{\min}+2)}}_{\Theta_{m,k_{\min}+2}} \underbrace{- \frac{(l+1)^2}{4(k_{\min}+2)}}_{\text{imaginary momentum factor}} = \frac{1}{2}. \tag{7.21}$$

On the other hand, if $m = (l+2)$, there are no weight $\frac{1}{2}$ states.

Therefore, we have seen that there are two NS-sector states of conformal weight $\frac{1}{2}$ for each $\hat{F}_l(\tau, z)$. Such states in the R sector must also be two. As is seen from their imaginary momentum factors, these four states have a common J_0^{tot} charge, and therefore the $J_0^{\text{tot}} = \tilde{J}_0^{\text{tot}}$ paring can be done as a supermultiplet. In the $A_{k_{\min}+1}$-type modular invariant theory, in which the holomorphic and anti-holomorphic combinations are (fully) diagonal, there are

$$(\mathbf{2}_{\text{NS}} \oplus \mathbf{2}_{\text{R}}) \otimes (\mathbf{2}_{\text{NS}} \oplus \mathbf{2}'_{\text{R}}) = \mathbf{8}_{\text{bosons}} \oplus \mathbf{8}_{\text{fermions}} \tag{7.22}$$

for each $l = 0, \ldots, k_{\min}$. If $\mathbf{2}_{\text{R}}$ and $\mathbf{2}'_{\text{R}}$ are the doublets of the same SU(2) factor of SO(4) (type IIB), the multiplet contains an anti-selfdual tensor. If, on the other hand, they are the different ones (type IIA), the multiplet is a vector multiplet.

This spectrum of massless states are precisely the ones expected from the geometry of the ALE spaces. This fact has already been anticipated in the analysis of [13]. They are opposite to the NS5-branes, and this observation is in agreement with the T-duality [7].

8. Summary and discussion

In this paper, we have considered type II and heterotic string compactifications on an isolated singularity in the noncompact Gepner model approach. We have mainly studied the threefold case, but also briefly discussed the twofold case. The conifold-type ADE singular Calabi-Yau threefolds are modeled by conformal field theory, which is a tensor product of the $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ Kazama-Suzuki model, an $N=2$ minimal model and a free conformal field theory describing the four-dimensional Minkowski space. We have used the result of [13] to construct new space-time supersymmetric, modular invariant partition functions for both type II and heterotic string theories, thereby the issue in the earlier noncompact Gepner models — the absence of the localized modes — has been resolved. We have investigated in detail the massless spectra of the localized modes. There are differences between when the level of the minimal model $k_{\min}$ is odd and when it is even. In particular, we found gapless spectra of continuous series representations in the even $k_{\min}$ case. The summary of massless spectra for various cases is shown in table 1. Among them, we have shown that the $k_{\min} = 3$ compactification of the $E_8 \times E_8$ heterotic string has three generations of matter fields in the $\mathbf{27} \oplus \mathbf{1}$ representation of E_6. They are not on an equal footing, and we propose that this model is worthy of further exploration as a viable alternative string model for the E_6 unification.

	Odd $k_{\min}$	Even $k_{\min}$
Type IIA	$\frac{k_{\min}+3}{2}$ hypermultiplets	$\frac{k_{\min}}{2}+1$ hypermultiplets (+gapless)
Type IIB	$\frac{k_{\min}+3}{2}$ vector multiplets	$\frac{k_{\min}}{2}+1$ vector multiplets (+gapless)
$E_8 \times E_8$ heterotic	$\frac{k_{\min}+3}{2}$ chiral supermultiplets in $\mathbf{10} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{16}$ of SO(10) (or $\mathbf{27} \oplus \mathbf{1}$ of E_6)	$\frac{k_{\min}}{2}+1$ chiral supermultiplets in $\mathbf{10} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{16}$ of SO(10) (or $\mathbf{27} \oplus \mathbf{1}$ of E_6) (+gapless)
SO(32) heterotic	$\frac{k_{\min}+3}{2}$ chiral supermultiplets in $\mathbf{26} \oplus \mathbf{1} \oplus \mathbf{1}$ of SO(26)	$\frac{k_{\min}}{2}+1$ chiral supermultiplets in $\mathbf{26} \oplus \mathbf{1} \oplus \mathbf{1}$ of SO(26) (+gapless)

Table 1: A summary of four-dimensional massless spectra for the $A_{k_{\min}+1}$ modular invariant model.

In the twofold case, we have confirmed in the type II case that the massless spectra of localized modes are consistent with the T-duality between the ALE spaces and the systems of NS5-branes. Although the heterotic cases have been omitted in this paper, the conversion can straightforwardly be done and will be reported in a future publication.

There are no localized gauge fields (nor localized gravity) in this model. If we interpret the Virasoro condition as the wave equation, as we usually do in critical string theories on a flat space-time, then the wave operator gets a mass term from the Liouville energy. However, we should note that, in a curved space, one cannot tell whether a field is massless or massive by looking only at the wave operator. A well-known example is the conformal mass in the AdS space [69]. Also, in a flat space with a linear-dilaton background, the scalar Laplacian in the Einstein frame gets a linear term in the derivative along the linear-dilaton direction. Therefore, we must be careful when we interpret the Liouville energy as the mass of the gauge fields or gravity. The decoupling of gravity and gauge fields from the localized modes may be regarded as a consequence of the assumption that the singularity is isolated. It would be interesting to explore the possibility of relaxing somehow this assumption (by, for instance, considering first a compact Calabi-Yau and tracing the gauge dynamics in the decoupling limit) so that their couplings may be discussed in the framework of conformal field theory.

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A. Theta functions and $N = 2$ minimal characters

In the appendices below, we assume that k is a positive integer.

Theta functions.

$$\Theta_{m,k}(\tau, z) \equiv \sum_{n\in\mathbf{Z}} q^{k(n+\frac{m}{2k})^2} y^{k(n+\frac{m}{2k})}, \quad q = e^{2\pi i\tau}, \quad y = e^{2\pi iz}, \tag{A.1}$$

where the level k is a positive integer, and m is an integer. It satisfies

$$\Theta_{m+2k,k}(\tau, z) = \Theta_{m,k}(\tau, z), \tag{A.2}$$

$$\Theta_{m,k}(\tau, -z) = \Theta_{-m,k}(\tau, z). \tag{A.3}$$

The Jacobi theta functions.

$$\vartheta_3(\tau, z) \equiv (\Theta_{0,2} + \Theta_{2,2})(\tau, z), \tag{A.4}$$

$$\vartheta_4(\tau, z) \equiv (\Theta_{0,2} - \Theta_{2,2})(\tau, z), \tag{A.5}$$

$$\vartheta_2(\tau, z) \equiv (\Theta_{1,2} + \Theta_{-1,2})(\tau, z), \tag{A.6}$$

$$\tilde{\vartheta}_1(\tau, z) \equiv (\Theta_{1,2} - \Theta_{-1,2})(\tau, z). \tag{A.7}$$

Here we have introduced the unconventional notation $\tilde{\vartheta}_1$ because it appears in the spectral flow orbit naturally rather than $\vartheta_1(\tau, z) = -i\tilde{\vartheta}_1(\tau, z)$.

The composition formula of theta functions.

$$\Theta_{m,k}(\tau, z)\Theta_{m',k'}(\tau, z') = \sum_{r\in\mathbf{Z}_{k+k'}} \Theta_{2rkk'+km'-k'm,kk'(k+k')}(\tau, u)\Theta_{2rk'+m+m',k+k'}(\tau, v) \tag{A.8}$$

$$\text{or} = \sum_{r\in\mathbf{Z}_{k+k'}} \Theta_{2rkk'-km'+k'm,kk'(k+k')}(\tau, -u)\Theta_{2rk+m+m',k+k'}(\tau, v), \tag{A.9}$$

$u = \frac{z'-z}{k+k'}$, $v = \frac{kz+k'z'}{k+k'}$.

The SU(2)$_k$ characters.

$$\chi_l^{(k)}(\tau, z) = \frac{\Theta_{l+1.k+2} - \Theta_{-l-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}}(\tau, z) \tag{A.10}$$

$$= \sum_{m\in\mathbf{Z}_{2k}} c_m^l(\tau)\Theta_{m,k}(\tau, z). \tag{A.11}$$

$l = 0, 1, \ldots, k$. The latter equation defines the string functions $c_m^l(\tau)$.

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Symmetries of the level-k string functions.

$$c^l_{m+2k}(\tau) = c^{k-l}_{m+k}(\tau) = c^l_m(\tau). \tag{A.12}$$

The $N = 2$ minimal characters.

$$\chi^{l,s}_m(\tau, z) = \sum_{r\in\mathbf{Z}_k} c^l_{m+4r-s}(\tau)\Theta_{2m+(k+2)(4r-s),2k(k+2)}(\tau, \frac{z}{k+2}), \tag{A.13}$$

$l = 0, 1, \ldots, k$, $m \in \mathbf{Z}_{2(k+2)}$, $s \in \mathbf{Z}_4$.

$$ch^{(\mathrm{NS})}_{l,m}(\tau, z) = (\chi^{l,0}_m + \chi^{l,2}_m)(\tau, z), \tag{A.14}$$

$$ch^{(\widetilde{\mathrm{NS}})}_{l,m}(\tau, z) = (\chi^{l,0}_m - \chi^{l,2}_m)(\tau, z), \tag{A.15}$$

$$ch^{(\mathrm{R})}_{l,m}(\tau, z) = (\chi^{l,1}_m + \chi^{l,-1}_m)(\tau, z), \tag{A.16}$$

$$ch^{(\widetilde{\mathrm{R}})}_{l,m}(\tau, z) = (\chi^{l,1}_m - \chi^{l,-1}_m)(\tau, z). \tag{A.17}$$

Symmetries of the $N = 2$ minimal characters.

$$\chi^{l,s}_{m+2(k+2)}(\tau, z) = \chi^{k-l,s+2}_{m+k+2}(\tau, z) = \chi^{l,s}_m(\tau, z). \tag{A.18}$$

An identity.

$$\chi^{(k)}_l(\tau, 0)\Theta_{s,2}(\tau, -z) = \sum_{m\in\mathbf{Z}_{2(k+2)}} \Theta_{m,k+2}\left(\tau, \frac{-2z}{k+2}\right)\chi^{l,s}_m(\tau, z), \tag{A.19}$$

which can be proved by using the composition formula:

$$\chi^{(k)}_l(\tau, z + u)\Theta_{s,2}(\tau, u) = \sum_{m\in\mathbf{Z}_{2(k+2)}} \chi^{l,s}_m(\tau, z)\Theta_{m,k+2}\left(\tau, u + \frac{kz}{k+2}\right). \tag{A.20}$$

Modular transformations.

$$\Theta_{m,k}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \sqrt{\frac{\tau}{2ik}}e^{\frac{\pi ikz^2}{2\tau}} \sum_{m'\in\mathbf{Z}_{2k}} e^{-\pi i\frac{mm'}{k}}\Theta_{m',k}(\tau, z), \tag{A.21}$$

$$\Theta_{m,k}(\tau + 1, z) = e^{\frac{\pi im^2}{2k}}\Theta_{m,k}(\tau, z). \tag{A.22}$$

B. Useful expressions of $F_{l,2r}(\tau, z)$ and $\hat{F}_{l,2r}(\tau, z)$

$F_{l,2r}(\tau, z)$

Let us name

$$\Theta_{(s,s')}(\tau, z) \equiv \sum_{\nu\in\mathbf{Z}_2} \Theta_{s+2\nu,2}(\tau, z)\Theta_{s'+2\nu,2}(\tau, z). \tag{B.1}$$

Then

$$\Theta_{(0,0)} = \Theta_{(2,2)} = \frac{\vartheta_3^2+\vartheta_4^2}{2}, \tag{B.2}$$

$$\Theta_{(0,2)} = \Theta_{(2,0)} = \frac{\vartheta_3^2-\vartheta_4^2}{2}, \tag{B.3}$$

$$\Theta_{(1,1)} = \Theta_{(-1,-1)} = \frac{\vartheta_2^2+\tilde{\vartheta}_1^2}{2}, \tag{B.4}$$

$$\Theta_{(1,-1)} = \Theta_{(-1,1)} = \frac{\vartheta_2^2-\tilde{\vartheta}_1^2}{2}. \tag{B.5}$$

$$\begin{aligned}\left(\Theta_{(0,0)}\Theta_{(0,2)} - \Theta_{(1,1)}\Theta_{(1,-1)}\right)(\tau,z) &= \frac{1}{4}\left(\vartheta_3^4-\vartheta_4^4-\vartheta_2^4+\tilde{\vartheta}_1^4\right)(\tau,z) \\ &= 0.\end{aligned} \tag{B.6}$$

Therefore, we can either write

$$\begin{aligned}&\frac{1}{4}\chi_l^{(k)}(\tau,0)\left(\vartheta_3^4-\vartheta_4^4-\vartheta_2^4+\tilde{\vartheta}_1^4\right)(\tau,z) \\ &\quad= \left(\chi_l^{(k)}(\tau,0)\Theta_{(0,0)}(\tau,z)\right)\Theta_{(0,2)}(\tau,z) - \left(\chi_l^{(k)}(\tau,0)\Theta_{(1,1)}(\tau,z)\right)\Theta_{(1,-1)}(\tau,z)\end{aligned} \tag{B.7}$$

and use the composition formula for theta functions in the parentheses first, or write

$$= \left(\chi_l^{(k)}(\tau,0)\Theta_{(0,2)}(\tau,z)\right)\Theta_{(0,0)}(\tau,z) - \left(\chi_l^{(k)}(\tau,0)\Theta_{(1,-1)}(\tau,z)\right)\Theta_{(1,1)}(\tau,z) \tag{B.8}$$

and do the same thing in this expression.

Let us compute (B.7) and (B.8) in two different ways. We first compute $\chi_l^{(k)}(\tau,0)\Theta_{(s,s')}(\tau,z)$. Since $\chi_l^{(k)}$ and $\Theta_{s,2}$ are composed into an $N=2$ minimal character and a level-$(k+2)$ theta function as shown in (A.20), we further combine this level-$(k+2)$ theta and the remaining level-2 theta in $\Theta_{(s,s')}(\tau,z)$ to find

$$\begin{aligned}\Theta_{-s',2}(\tau,-u)\Theta_{m,k+2}\left(\tau,u+\frac{kz}{k+2}\right) &= \sum_{r\in\mathbf{Z}_{k+4}}\Theta_{-(k+2)s'-2m+4(k+2)r,2(k+2)(k+4)}\left(\tau,-\frac{2u+\frac{kz}{k+2}}{k+4}\right) \\ &\quad\cdot\Theta_{-s'+m+4r,k+4}\left(\tau,\frac{k(z+u)}{k+4}\right),\end{aligned} \tag{B.9}$$

$$\begin{aligned}\chi_l^{(k)}(\tau,0)\Theta_{(s,s')}(\tau,z) &= \sum_{r\in\mathbf{Z}_{k+4}+\frac{l+s+s'}{2}}\ \sum_{m\in\mathbf{Z}_{4(k+2)}}\delta_{m,l+s}^{(\mathrm{mod}2)}\chi_m^{l,m-(2r+s-s')}(\tau,z) \\ &\quad\cdot\Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+2}\right)\Theta_{2r,k+4}(\tau,0).\end{aligned} \tag{B.10}$$

Using (B.10), we find

$$\begin{aligned}(B.7) = &\sum_{r\in\mathbf{Z}_{k+4}+\frac{l}{2}}\Theta_{2r,k+4}(\tau,0)\sum_{m\in\mathbf{Z}_{4(k+2)}}\left(\delta_{m,l}^{(\mathrm{mod}2)}\Theta_{(0,2)}(\tau,z)-\delta_{m,l+1}^{(\mathrm{mod}2)}\Theta_{(1,-1)}(\tau,z)\right) \\ &\cdot\chi_m^{l,m-2r}(\tau,z)\Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+2}\right)\end{aligned}$$

$$
\begin{aligned}
&\equiv \sum_{r\in\mathbf{Z}_{k+4}+\frac{l}{2}} \Theta_{2r,k+4}(\tau,0)F^{(-)}_{l,2r}(\tau,z), &\text{(B.11)}\\
&= \sum_{r\in\mathbf{Z}_{k+4}+\frac{l}{2}} \Theta_{2r,k+4}(\tau,0)\sum_{m\in\mathbf{Z}_{4(k+2)}}\left(\delta^{(\mathrm{mod}2)}_{m,l}\Theta_{(0,0)}(\tau,z)-\delta^{(\mathrm{mod}2)}_{m,l+1}\Theta_{(1,1)}(\tau,z)\right)\\
&\qquad\cdot\chi^{l,m-2r+2}_{m}(\tau,z)\Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+2}\right)\\
&\equiv \sum_{r\in\mathbf{Z}_{k+4}+\frac{l}{2}} \Theta_{2r,k+4}(\tau,0)F^{(+)}_{l,2r}(\tau,z). &\text{(B.12)}
\end{aligned}
$$

On the other hand, if we apply the composition formula to the two theta functions in $\Theta_{(s,s')}$, we find

$$
\Theta_{(s,s')}(\tau,z) = \sum_{\nu\in\mathbf{Z}_2}\sum_{t\in\mathbf{Z}_4}\Theta_{8t+8\nu+2s+2s',16}\left(\tau,\frac{z}{2}\right)\Theta_{4t-s+s',4}(\tau,0), \tag{B.13}
$$

$$
\begin{aligned}
\chi^{(k)}_l(\tau,0)\Theta_{(s,s')}(\tau,z) &= \sum_{m\in\mathbf{Z}_{2k}} c^l_m(\tau)\Theta_{m,k}(\tau,0)\sum_{\nu\in\mathbf{Z}_2}\sum_{t\in\mathbf{Z}_4}\Theta_{8t+8\nu+2s+2s',16}\left(\tau,\frac{z}{2}\right)\Theta_{4t-s+s',4}(\tau,0)\\
&= \sum_{r'\in\mathbf{Z}_{2(k+4)}}\Theta_{r'k+4}(\tau,0)\sum_{m\in\mathbf{Z}_{2k}} c^l_{r'+4m+s-s'}(\tau)\Theta_{\frac{s+s'}{2},1}(\tau,2z)\\
&\qquad\cdot\Theta_{-4r'+(k+4)(-4m-s+s'),4k(k+4)}(\tau,0). \qquad\text{(B.14)}
\end{aligned}
$$

This way of composition of theta functions leads to different expressions of (B.7) and (B.8):

$$
\begin{aligned}
(B.7) &= \sum_{\mathbf{Z}_{k+4}+\frac{l}{2}}\Theta_{2r,k+4}(\tau,0)\sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m}(\tau)\Theta_{-8r-(k+4)4m,4k(k+4)}(\tau,0)\\
&\quad\cdot(\Theta_{0,1}(\tau,2z)\Theta_{(0,2)}(\tau,z)-\Theta_{1,1}(\tau,2z)\Theta_{(1,-1)}(\tau,z)), &\text{(B.15)}
\end{aligned}
$$

$$
\begin{aligned}
(B.8) &= \sum_{\mathbf{Z}_{k+4}+\frac{l}{2}}\Theta_{2r,k+4}(\tau,0)\sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m-2}(\tau)\Theta_{-8r-(k+4)(4m-2),4k(k+4)}(\tau,0)\\
&\quad\cdot(\Theta_{1,1}(\tau,2z)\Theta_{(0,0)}(\tau,z)-\Theta_{0,1}(\tau,2z)\Theta_{(1,1)}(\tau,z)). &\text{(B.16)}
\end{aligned}
$$

Thus we find, for $r\in\mathbf{Z}_{k+4}+\frac{l}{2}$,

$$
\begin{aligned}
F^{(-)}_{l,2r}(\tau,z) &= \sum_{m\in\mathbf{Z}_{4(k+2)}}\chi^{l,m-2r}_m(\tau,z)\Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+2}\right)\\
&\qquad\cdot\left(\delta^{(\mathrm{mod}2)}_{m,l}\Theta_{(0,2)}(\tau,z)-\delta^{(\mathrm{mod}2)}_{m,l+1}\Theta_{(1,-1)}(\tau,z)\right) &\text{(B.17)}\\
&= \sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m}(\tau)\Theta_{-8r-(k+4)4m,4k(k+4)}(\tau,0)\\
&\qquad\cdot(\Theta_{0,1}(\tau,2z)\Theta_{(0,2)}(\tau,z)-\Theta_{1,1}(\tau,2z)\Theta_{(1,-1)}(\tau,z))\\
&= \frac{1}{2}\sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m}(\tau)\Theta_{-8r-(k+4)4m,4k(k+4)}(\tau,0)\Lambda_2(\tau,z), &\text{(B.18)}
\end{aligned}
$$

$$
F^{(+)}_{l,2r}(\tau,z) = \sum_{m\in\mathbf{Z}_{4(k+2)}}\chi^{l,m-2r+2}_m(\tau,z)\Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+2}\right)
$$

$$\cdot\left(\delta^{(\mathrm{mod}2)}_{m,l}\Theta_{(0,0)}(\tau,z)-\delta^{(\mathrm{mod}2)}_{m,l+1}\Theta_{(1,1)}(\tau,z)\right) \tag{B.19}$$

$$=\sum_{m\in\mathbf{Z}_{2k}} c^{l}_{2r+4m-2}(\tau)\Theta_{-8r-(k+4)(4m-2),4k(k+4)}(\tau,0)$$

$$\cdot\left(\Theta_{1,1}(\tau,2z)\Theta_{(0,0)}(\tau,z)-\Theta_{0,1}(\tau,2z)\Theta_{(1,1)}(\tau,z)\right)$$

$$=\frac{1}{2}\sum_{m\in\mathbf{Z}_{2k}} c^{l}_{2r+4m-2}(\tau)\Theta_{-8r-(k+4)(4m-2),4k(k+4)}(\tau,0)\Lambda_1(\tau,z), \tag{B.20}$$

where

$$\Lambda_1(\tau,z)\equiv 2\left(\Theta_{1,1}(\tau,2z)\Theta_{(0,0)}(\tau,z)-\Theta_{0,1}(\tau,2z)\Theta_{(1,1)}(\tau,z)\right), \tag{B.21}$$

$$\Lambda_2(\tau,z)\equiv 2\left(\Theta_{0,1}(\tau,2z)\Theta_{(0,2)}(\tau,z)-\Theta_{1,1}(\tau,2z)\Theta_{(1,-1)}(\tau,z)\right) \tag{B.22}$$

are the same as (3.7), (3.8) in the text. (The definition of $\Theta_{(s,s')}(\tau,z)$ is given at the beginning of this appendix.) In particular, even if $k=0$, the equation (A.20) still holds if we define

$$\chi^{l=0,s}_{m}(\tau,z)\equiv\delta^{(\mathrm{mod}4)}_{m,s}\quad(m,s\in\mathbf{Z}_4), \tag{B.23}$$

then we have

$$F^{(-)}_{l,2r}(\tau,z)=\begin{cases}\frac{1}{2}\Lambda_2(\tau,z) & \text{if } r=0,2,\\ 0 & \text{if } r=1,3,\end{cases} \tag{B.24}$$

$$F^{(+)}_{l,2r}(\tau,z)=\begin{cases}0 & \text{if } r=0,2,\\ \frac{1}{2}\Lambda_1(\tau,z) & \text{if } r=1,3.\end{cases} \tag{B.25}$$

The total $F_{l,2r}(\tau,z)$ function (3.24) is given by

$$F_{l,2r}(\tau,z)=\frac{1}{2}\left(F^{(-)}_{l,2r}(\tau,z)+F^{(+)}_{l,2r}(\tau,z)\right)$$

$$=\frac{1}{2}\sum_{\nu\in\mathbf{Z}_{4(k_{\min}+2)}}\sum_{\substack{\nu_0,\nu_1,\nu_2\in\mathbf{Z}_2\\ \nu_0+\nu_1+\nu_2\\ \equiv 1(\mathrm{mod}2)}}(-1)^{\nu}\chi^{l,l-2r+2\nu_0+\nu}_{l+\nu}(\tau,z)\Theta_{2\nu_1+\nu,2}(\tau,z)\Theta_{2\nu_2+\nu,2}(\tau,z)$$

$$\cdot\,\Theta_{(k_{\min}+2)2r-(k_{\min}+4)(l+\nu),2(k_{\min}+2)(k_{\min}+4)}\left(\tau,\frac{z}{k_{\min}+2}\right). \tag{B.26}$$

$F_{l,2r}(\tau,z)$ satisfies

$$F_{l,2r+2(k+4)}(\tau,z)=F_{l,2r}(\tau,z) \tag{B.27}$$

which is obvious due to the periodicity of theta functions. Also it is easy to show that [9]

$$F_{k-l,2r+k+4}(\tau,z)=F_{l,2r}(\tau,z). \tag{B.28}$$

$\hat{F}_{l,2r}(\tau,z)$

$\hat{F}_{l,2r}(\tau,z)$ functions (4.23) can be obtained by modifying the z-dependences of various theta functions as

$$\chi_m^{l,s}(\tau,z) \to \chi_m^{l,s}(\tau,0), \tag{B.29}$$

$$(\Theta_{(s,s')}(\tau,z) \equiv \ ) \sum_{\nu\in\mathbf{Z}_2} \Theta_{s+2\nu,2}(\tau,z)\Theta_{s'+2\nu,2}(\tau,z) \to \sum_{\nu\in\mathbf{Z}_2} \Theta_{s+2\nu,2}(\tau,0)\Theta_{s'+2\nu,2}(\tau,z), \tag{B.30}$$

$$\Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+2}\right) \to \Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+4}\right) \tag{B.31}$$

in $F^{(\pm)}_{l,2r}(\tau,z)$. The following formulas are useful:

$$\begin{aligned}
\hat{F}^{(-)}_{l,2r}(\tau,z) \equiv& \sum_{m\in\mathbf{Z}_{4(k+2)}} \chi_m^{l,m-2r}(\tau,0)\Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+4}\right)\\
&\qquad\cdot\left(\delta^{(\mathrm{mod}2)}_{m,l}\Theta_{(0,2)}(\tau;0,z) - \delta^{(\mathrm{mod}2)}_{m,l+1}\Theta_{(1,-1)}(\tau;0,z)\right) \qquad (B.32)\\
=& \sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m}(\tau)\Theta_{-8r-(k+4)4m,4k(k+4)}\left(\tau,\frac{z}{2(k+4)}\right)\\
&\qquad\cdot\left(\Theta_{0,1}(\tau,z)\Theta_{(0,2)}(\tau;0,z) - \Theta_{1,1}(\tau,z)\Theta_{(1,-1)}(\tau;0,z)\right)\\
=& \frac{1}{2}\sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m}(\tau)\Theta_{-8r-(k+4)4m,4k(k+4)}\left(\tau,\frac{z}{2(k+4)}\right)\hat{\Lambda}_2(\tau,z), \qquad (B.33)
\end{aligned}$$

$$\begin{aligned}
\hat{F}^{(+)}_{l,2r}(\tau,z) \equiv& \sum_{m\in\mathbf{Z}_{4(k+2)}} \chi_m^{l,m-2r+2}(\tau,0)\Theta_{(k+2)2r-(k+4)m,2(k+2)(k+4)}\left(\tau,\frac{z}{k+4}\right)\\
&\qquad\cdot\left(\delta^{(\mathrm{mod}2)}_{m,l}\Theta_{(0,0)}(\tau;0,z) - \delta^{(\mathrm{mod}2)}_{m,l+1}\Theta_{(1,1)}(\tau;0,z)\right) \qquad (B.34)\\
=& \sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m-2}(\tau)\Theta_{-8r-(k+4)(4m-2),4k(k+4)}\left(\tau,\frac{z}{2(k+4)}\right)\\
&\qquad\cdot\left(\Theta_{1,1}(\tau,z)\Theta_{(0,0)}(\tau;0,z) - \Theta_{0,1}(\tau,z)\Theta_{(1,1)}(\tau;0,z)\right)\\
=& \frac{1}{2}\sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m-2}(\tau)\Theta_{-8r-(k+4)(4m-2),4k(k+4)}\left(\tau,\frac{z}{2(k+4)}\right)\hat{\Lambda}_1(\tau,z),
\end{aligned} \tag{B.35}$$

$$\hat{F}_{l,2r}(\tau,z) = \frac{1}{2}\left(\hat{F}^{(-)}_{l,2r}(\tau,z) + \hat{F}^{(+)}_{l,2r}(\tau,z)\right), \tag{B.36}$$

where

$$\hat{\Lambda}_1(\tau,z) = 2\left(\Theta_{1,1}(\tau,z)\Theta_{(0,0)}(\tau;0,z) - \Theta_{0,1}(\tau,z)\Theta_{(1,1)}(\tau;0,z)\right), \tag{B.37}$$

$$\hat{\Lambda}_2(\tau,z) = 2\left(\Theta_{0,1}(\tau,z)\Theta_{(0,2)}(\tau;0,z) - \Theta_{1,1}(\tau,z)\Theta_{(1,-1)}(\tau;0,z)\right), \tag{B.38}$$

$$\Theta_{(s,s')}(\tau;z,z') \equiv \sum_{\nu\in\mathbf{Z}_2} \Theta_{s+2\nu,2}(\tau,z)\Theta_{s'+2\nu,2}(\tau,z'). \tag{B.39}$$

The expressions (B.37) and (B.38) are equivalent to the definitions (4.6) and (4.7) in the text. (Note, again, that k here is $k_{\min}$ in (4.23).)

$\hat{F}_{l,2r}(\tau, z)$ also satisfies

$$\hat{F}_{l,2r+2(k+4)}(\tau, z) = \hat{F}_{l,2r}(\tau, z), \tag{B.40}$$
$$\hat{F}_{k-l,2r+k+4}(\tau, z) = \hat{F}_{l,2r}(\tau, z), \tag{B.41}$$
$$\hat{F}_{l,-2r}(\tau, z) = \hat{F}_{l,2r}(\tau, -z). \tag{B.42}$$

C. Heterotic conversion

In Gepner models, any modular invariant partition function for type II strings can be converted to that for heterotic strings by a straightforward procedure [5], which we review in this appendix.

Let us denote level-1 affine SO($2n$) characters by

$$B_0^{(2n)}(\tau, z) \equiv \frac{(\vartheta_3(\tau, z))^n + (\vartheta_4(\tau, z))^n}{2(\eta(\tau))^n}, \tag{C.1}$$
$$B_v^{(2n)}(\tau, z) \equiv \frac{(\vartheta_3(\tau, z))^n - (\vartheta_4(\tau, z))^n}{2(\eta(\tau))^n}, \tag{C.2}$$
$$B_s^{(2n)}(\tau, z) \equiv \frac{(\vartheta_2(\tau, z))^n + (\tilde{\vartheta}_1(\tau, z))^n}{2(\eta(\tau))^n}, \tag{C.3}$$
$$B_{\bar{s}}^{(2n)}(\tau, z) \equiv \frac{(\vartheta_2(\tau, z))^n - (\tilde{\vartheta}_1(\tau, z))^n}{2(\eta(\tau))^n}. \tag{C.4}$$

Writing them as a column vector $\mathbf{B}^{(2n)}(\tau, z)$, their modular S- and T-transformations are given in the matrix notation

$$\mathbf{B}^{(2n)}(\tau, z)\Big|_S = e^{\frac{n\pi i z^2}{\tau}} S^{(2n)} \mathbf{B}^{(2n)}(\tau, z), \tag{C.5}$$
$$S^{(2n)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-n} & -i^{-n} \\ 1 & -1 & -i^{-n} & i^{-n} \end{pmatrix} \tag{C.6}$$

and

$$\mathbf{B}^{(2n)}(\tau, z)\Big|_T = T^{(2n)} \mathbf{B}^{(2n)}(\tau, z), \tag{C.7}$$
$$T^{(2n)} = \begin{pmatrix} e^{-\frac{n\pi i}{12}} & & & \\ & -e^{-\frac{n\pi i}{12}} & & \\ & & e^{+\frac{n\pi i}{6}} & \\ & & & e^{+\frac{n\pi i}{6}} \end{pmatrix}. \tag{C.8}$$

We also define

$$B^{(E_8)}(\tau, z) \equiv \frac{(\vartheta_3(\tau, z))^8 + (\vartheta_4(\tau, z))^8 + (\vartheta_2(\tau, z))^8 + (\tilde{\vartheta}_1(\tau, z))^8}{2(\eta(\tau))^8}, \tag{C.9}$$

then

$$
\begin{aligned}
B^{(E_8)}(\tau,z)\Big|_S &= e^{\frac{8\pi i z^2}{\tau}} B^{(E_8)}(\tau,z), && \text{(C.10)}\\
B^{(E_8)}(\tau,z)\Big|_T &= e^{-\frac{8\pi i}{12}} B^{(E_8)}(\tau,z). && \text{(C.11)}
\end{aligned}
$$

If we set $z = 0$, then we find

$$
\begin{aligned}
S^{(d+8)} &= S^{(d+24)} = S^{(d)} && = M^T S^{(d)} M, && \text{(C.12)}\\
e^{-\frac{8}{12}\pi i}\, T^{(d+8)} &= T^{(d+24)} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} T^{(d)} && = M^T T^{(d)} M, && \text{(C.13)}
\end{aligned}
$$

where

$$
M = M^T = M^{-1} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{C.14}
$$

d is the transverse space dimensions (that is, $d = 2$ for a four-dimensional flat Minkowski spacetime with a Calabi-Yau threefold, and $d = 4$ for six-dimensional one with a Calabi-Yau twofold). Therefore, $MB^{(E_8)}\mathbf{B}^{(d+8)}(\tau,0)$ and $M\mathbf{B}^{(d+24)}(\tau,0)$ transform exactly in the same manner as $\mathbf{B}^{(d)}(\tau,0)$ does under the modular S- and T-transformations. This means that starting from any modular invariant partition function for type II strings, we can obtain one for the $E_8 \times E_8$ heterotic string theory by replacing the left-moving fermion theta functions as (with all z's being equal to zero)

$$
\begin{aligned}
\frac{(\vartheta_3)^{\frac{d}{2}} + (\vartheta_4)^{\frac{d}{2}}}{2\eta^{\frac{d}{2}}}\ (= B_0^{(d)}) &\to \frac{(\vartheta_3)^{\frac{d+8}{2}} - (\vartheta_4)^{\frac{d+8}{2}}}{2\eta^{\frac{d+8}{2}}} B^{(E_8)}\ (= B_v^{(d+8)} B^{(E_8)}), && \text{(C.15)}\\
\frac{(\vartheta_3)^{\frac{d}{2}} - (\vartheta_4)^{\frac{d}{2}}}{2\eta^{\frac{d}{2}}}\ (= B_v^{(d)}) &\to \frac{(\vartheta_3)^{\frac{d+8}{2}} + (\vartheta_4)^{\frac{d+8}{2}}}{2\eta^{\frac{d+8}{2}}} B^{(E_8)},\ (= B_0^{(d+8)} B^{(E_8)}), && \text{(C.16)}\\
\frac{(\vartheta_2)^{\frac{d}{2}} + (\tilde\vartheta_1)^{\frac{d}{2}}}{2\eta^{\frac{d}{2}}}\ (= B_s^{(d)}) &\to -\frac{(\vartheta_2)^{\frac{d+8}{2}} + (\tilde\vartheta_1)^{\frac{d+8}{2}}}{2\eta^{\frac{d+8}{2}}} B^{(E_8)},\ (= -B_s^{(d+8)} B^{(E_8)}), && \text{(C.17)}\\
\frac{(\vartheta_2)^{\frac{d}{2}} - (\tilde\vartheta_1)^{\frac{d}{2}}}{2\eta^{\frac{d}{2}}}\ (= B_{\bar s}^{(d)}) &\to -\frac{(\vartheta_2)^{\frac{d+8}{2}} - (\tilde\vartheta_1)^{\frac{d+8}{2}}}{2\eta^{\frac{d+8}{2}}} B^{(E_8)}\ (= -B_{\bar s}^{(d+8)} B^{(E_8)}) && \text{(C.18)}
\end{aligned}
$$

and also for the SO(32) theory as

$$
\begin{aligned}
\frac{(\vartheta_3)^{\frac{d}{2}} + (\vartheta_4)^{\frac{d}{2}}}{2\eta^{\frac{d}{2}}}\ (= B_0^{(d)}) &\to \frac{(\vartheta_3)^{\frac{d+24}{2}} - (\vartheta_4)^{\frac{d+24}{2}}}{2\eta^{\frac{d+24}{2}}},\ (= B_v^{(d+24)}) && \text{(C.19)}\\
\frac{(\vartheta_3)^{\frac{d}{2}} - (\vartheta_4)^{\frac{d}{2}}}{2\eta^{\frac{d}{2}}}\ (= B_v^{(d)}) &\to \frac{(\vartheta_3)^{\frac{d+24}{2}} + (\vartheta_4)^{\frac{d+24}{2}}}{2\eta^{\frac{d+24}{2}}},\ (= B_0^{(d+24)}) && \text{(C.20)}
\end{aligned}
$$

$$\frac{(\vartheta_2)^{\frac{d}{2}}+(\tilde{\vartheta}_1)^{\frac{d}{2}}}{2\eta^{\frac{d}{2}}}\ (=B_s^{(d)}) \to -\frac{(\vartheta_2)^{\frac{d+24}{2}}+(\tilde{\vartheta}_1)^{\frac{d+24}{2}}}{2\eta^{\frac{d+24}{2}}},\ (=-B_s^{(d+24)}) \tag{C.21}$$

$$\frac{(\vartheta_2)^{\frac{d}{2}}-(\tilde{\vartheta}_1)^{\frac{d}{2}}}{2\eta^{\frac{d}{2}}}\ (=B_{\bar{s}}^{(d)}) \to -\frac{(\vartheta_2)^{\frac{d+24}{2}}-(\tilde{\vartheta}_1)^{\frac{d+24}{2}}}{2\eta^{\frac{d+24}{2}}}\ (=-B_{\bar{s}}^{(d+24)}). \tag{C.22}$$

Applying these rules in $\hat{F}_{l,2r}(\tau,z)$, we obtain

$$\hat{F}_{l,2r}^{E_8\times E_8}(\tau,z) = \frac{1}{2}\left(\hat{F}_{l,2r}^{E_8\times E_8(-)}(\tau,z)+\hat{F}_{l,2r}^{E_8\times E_8(+)}(\tau,z)\right), \tag{C.23}$$

$$\hat{F}_{l,2r}^{E_8\times E_8(-)}(\tau,z) \equiv \sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m}(\tau)\Theta_{-8r-(k+4)4m,4k(k+4)}\left(\tau,\frac{z}{2(k+4)}\right)\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8}(\tau,z), \tag{C.24}$$

$$\hat{F}_{l,2r}^{E_8\times E_8(+)}(\tau,z) \equiv \sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m-2}(\tau)\Theta_{-8r-(k+4)(4m-2),4k(k+4)}\left(\tau,\frac{z}{2(k+4)}\right)\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8}(\tau,z), \tag{C.25}$$

$$\begin{aligned}\frac{\frac{1}{2}\hat{\Lambda}_1^{E_8\times E_8}(\tau,z)}{\eta^{14}(\tau)} \equiv &\Big(\Theta_{1,1}(\tau,z)\left(B_v^{(10)}(\tau,0)B_0^{(2)}(\tau,z)+B_0^{(10)}(\tau,0)B_v^{(2)}(\tau,z)\right)\\ &+\Theta_{0,1}(\tau,z)\left(B_s^{(10)}(\tau,0)B_s^{(2)}(\tau,z)+B_{\bar{s}}^{(10)}(\tau,0)B_{\bar{s}}^{(2)}(\tau,z)\right)\Big)B^{(E_8)}(\tau,0),\end{aligned} \tag{C.26}$$

$$\begin{aligned}\frac{\frac{1}{2}\hat{\Lambda}_2^{E_8\times E_8}(\tau,z)}{\eta^{14}(\tau)} \equiv &\Big(\Theta_{0,1}(\tau,z)\left(B_0^{(10)}(\tau,0)B_0^{(2)}(\tau,z)+B_v^{(10)}(\tau,0)B_v^{(2)}(\tau,z)\right)\\ &+\Theta_{1,1}(\tau,z)\left(B_s^{(10)}(\tau,0)B_{\bar{s}}^{(2)}(\tau,z)+B_{\bar{s}}^{(10)}(\tau,0)B_s^{(2)}(\tau,z)\right)\Big)B^{(E_8)}(\tau,0),\end{aligned} \tag{C.27}$$

and

$$\hat{F}_{l,2r}^{\mathrm{SO}(32)}(\tau,z) = \frac{1}{2}\left(\hat{F}_{l,2r}^{\mathrm{SO}(32)(-)}(\tau,z)+\hat{F}_{l,2r}^{\mathrm{SO}(32)(+)}(\tau,z)\right), \tag{C.28}$$

$$\hat{F}_{l,2r}^{\mathrm{SO}(32)(-)}(\tau,z) \equiv \sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m}(\tau)\Theta_{-8r-(k+4)4m,4k(k+4)}\left(\tau,\frac{z}{2(k+4)}\right)\frac{1}{2}\hat{\Lambda}_2^{\mathrm{SO}(32)}(\tau,z), \tag{C.29}$$

$$\hat{F}_{l,2r}^{\mathrm{SO}(32)(+)}(\tau,z) \equiv \sum_{m\in\mathbf{Z}_{2k}} c^l_{2r+4m-2}(\tau)\Theta_{-8r-(k+4)(4m-2),4k(k+4)}\left(\tau,\frac{z}{2(k+4)}\right)\frac{1}{2}\hat{\Lambda}_1^{\mathrm{SO}(32)}(\tau,z), \tag{C.30}$$

$$\begin{aligned}\frac{\frac{1}{2}\hat{\Lambda}_1^{\mathrm{SO}(32)}(\tau,z)}{\eta^{14}(\tau)} \equiv &\ \Theta_{1,1}(\tau,z)\left(B_v^{(26)}(\tau,0)B_0^{(2)}(\tau,z)+B_0^{(26)}(\tau,0)B_v^{(2)}(\tau,z)\right)\\ &+\Theta_{0,1}(\tau,z)\left(B_s^{(26)}(\tau,0)B_s^{(2)}(\tau,z)+B_{\bar{s}}^{(26)}(\tau,0)B_{\bar{s}}^{(2)}(\tau,z)\right),\end{aligned} \tag{C.31}$$

$$\begin{aligned}\frac{\frac{1}{2}\hat{\Lambda}_2^{\mathrm{SO}(32)}(\tau,z)}{\eta^{14}(\tau)} \equiv &\ \Theta_{0,1}(\tau,z)\left(B_0^{(26)}(\tau,0)B_0^{(2)}(\tau,z)+B_v^{(26)}(\tau,0)B_v^{(2)}(\tau,z)\right)\\ &+\Theta_{1,1}(\tau,z)\left(B_s^{(26)}(\tau,0)B_{\bar{s}}^{(2)}(\tau,z)+B_{\bar{s}}^{(26)}(\tau,0)B_s^{(2)}(\tau,z)\right).\end{aligned} \tag{C.32}$$

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D. Proof of the regularization formula

In this appendix we prove the regularization formula. Let

$$f(z,\epsilon) \equiv -\sum_{n=0}^{\infty} \frac{e^{-n\epsilon}}{z-n}, \tag{D.1}$$

then $f(z,\epsilon)$ has simple poles at $z=n$, $(n=0,1,2,\ldots)$ with residue $-e^{-n\epsilon}$. On the other hand, the gamma function $\Gamma(-z)$ has also simple poles at $z=n$, $(n=0,1,2,\ldots)$, and so do $\frac{\partial}{\partial z}\log(\Gamma(-z))$ at the same locations with residue -1. Therefore, comparing the singularities, we may write

$$\frac{\partial^2}{\partial z^2}\log(\Gamma(-z)) = \sum_{n=0}^{\infty} \frac{+1}{(z-n)^2} + \text{const.} \tag{D.2}$$

Subtracting

$$\frac{\partial}{\partial z} f(z,\epsilon) = \sum_{n=0}^{\infty} \frac{e^{-n\epsilon}}{(z-n)^2} \tag{D.3}$$

from both sides and integrating with respect to z, we find

$$\frac{\partial}{\partial z}\log\Gamma(-z) - f(z,\epsilon) = -\sum_{n=0}^{\infty} \frac{1-e^{-n\epsilon}}{z-n} + az + b \tag{D.4}$$

for some constants a and b. The first term is $O(\epsilon)$.

To determine a and b, we set $z=-1$ and $z=2$:

$$f(-1,\epsilon) = -e^{\epsilon}\log(1-e^{-\epsilon}), \tag{D.5}$$
$$f(-2,\epsilon) = -e^{2\epsilon}\log(1-e^{-\epsilon}) - e^{\epsilon}, \tag{D.6}$$

and therefore

$$-e^{\epsilon}\log(1-e^{-\epsilon}) = \frac{\partial}{\partial z}\log\Gamma(-z)\bigg|_{z=-1} + O(\epsilon) - (-a+b), \tag{D.7}$$
$$-e^{2\epsilon}\log(1-e^{-\epsilon}) - e^{\epsilon} = \frac{\partial}{\partial z}\log\Gamma(-z)\bigg|_{z=-2} + O(\epsilon) - (-2a+b). \tag{D.8}$$

$$\psi(z) \equiv \frac{\partial}{\partial z}\log\Gamma(z) \tag{D.9}$$

is known as the psi-function (or the $d\Gamma$-function), and

$$\frac{\partial}{\partial z}\log\Gamma(-z) = -\psi(-z). \tag{D.10}$$

The psi-function satisfies the recursion relation

$$\psi(z+1) = \frac{1}{z} + \psi(z), \tag{D.11}$$

and hence

$$\psi(2) = \psi(1) + 1 \\ = -\mathcal{C} + 1, \tag{D.12}$$

where

$$-\psi(1) = \frac{\partial}{\partial z} \log \Gamma(-z)\bigg|_{z=-1} = \mathcal{C} \tag{D.13}$$

is known as Euler's constant. Using these data, we find

$$a = O(\epsilon), \tag{D.14}$$

$$b = \log \epsilon + \mathcal{C} + O(\epsilon) + O(\epsilon \log \epsilon), \tag{D.15}$$

and obtain the final form of the regularization formula

$$-\sum_{n=0}^{\infty} \frac{e^{-n\epsilon}}{z-n} = -\log \epsilon + \frac{\partial}{\partial z} \log \Gamma(-z) - \mathcal{C} + O(\epsilon) + O(\epsilon \log \epsilon). \tag{D.16}$$

That this formula is correct can also be confirmed numerically by Mathematica.

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